Presented at SSD99

Generalizing Graphs using Amalgamation and Selection

J. G. Stell and M. F. Worboys

Department of Computer Science, Keele University
Keele, Staffs, ST5 5BG, U. K.
{john,michael}@cs.keele.ac.uk

Abstract. This work is a contribution to the developing literature on multi-resolution data models. It considers operations for model-oriented generalization in the case where the underlying data is structured as a graph. The paper presents a new approach in that a distinction is made between generalizations that amalgamate data objects and those that select data objects. We show that these two types of generalization are conceptually distinct, and provide a formal framework in which both can be understood. Generalizations that are combinations of amalgamation and selection are termed simplifications, and the paper provides a formal framework in which simplifications can be computed (for example, as compositions of other simplifications). A detailed case study is presented to illustrate the techniques developed, and directions for further work are discussed.

1 Introduction

Specialist spatial information systems (SIS) play an increasingly important role within the Information Technology industry [Abe97]. For the potential of SIS to be fully realised, spatial database functionality needs to be integrated with other more generic aspects of database technology. Spatial data comprise a valuable subset of the totality of data holdings of an enterprise and their utility is optimized when they are flexible enough to be capable of integration with other data sets in a variety of ways. The focus of this paper is a contribution towards the provision of flexibility with regard to the scale or resolution at which data are handled. Resolution is concerned with the level of discernibility between elements of a phenomenon that is being represented by the data, and higher resolutions allow more detail to be observed in the components of the phenomenon. Flexibility in handling resolution is advanced by provision of multi-resolution data models, where data are managed in the SIS at a variety of levels of detail.

The issue of multi-resolution spatial datasets has been taken up by several authors (e.g. [PD95,RS95]). In our own earlier work [SW98,Wor98a,Wor98b] we proposed a general model that helps to provide a formal basis for processing and reasoning with spatial data that are heterogeneous with regard to semantic and geometric precision. For multi-resolution data models to be effective, there must be the means to make appropriate transitions between levels of detail in the data. Transition from higher to lower resolutions is often referred to as *generalization*. Cartographic generalization has been the subject of a great deal of research by the cartographic and GIS communities, particularly on the geometric aspects of the generalization process (see for example [BM91,MLW95]). When the word 'generalization' is used in this paper it usually refers to model-oriented generalization, in the sense of [M$^+$95], as we are not concerned here with the particular form of the cartographic representation of the data.

The focus of this paper is on the geometric components of geospatial data. In particular, the emphasis here is on network data structures, as they provide a simpler case than fully two-dimensional data structures, and yet have many applications to real world systems. The paper seeks to make a clear formal distinction between model-oriented generalizations that are based on selection of data and those that are based on data amalgamation. This is a distinction that is somewhat blurred in some of the earlier multi-resolution spatial data models.

In the next section, the background to this research is outlined, particularly in the context of multi-resolution data models, generalization and functionality in databases for handling graphs. The following section makes precise the distinction between selection and amalgamation transformations from higher to lower levels of detail. The remainder of the paper is devoted to working out in detail the formal properties of selection and amalgamation operations on graphs, and includes consideration of a detailed case study.

2 Background

2.1 Multi-resolution Data Models and Generalization

Generalization is the process of transforming a representation of a geographic space to a less detailed one. The representation may be in terms of a data/process model, in which case the transformation is called *model generalization*, or involve visualization of the space on an output device or hard copy, in which case the transformation is called *cartographic generalization*. Cartographic generalization has been the subject of a great deal of research by the cartographic and GIS communities, particularly on the geometric aspects of the generalization process (see for example [BM91,MLW95]). Model-oriented generalization was introduced by Müller *et al.* [M$^+$95]. Rigaux and Scholl [RS95] discuss the impact of scale and resolution on spatial data modelling and querying. They develop a theory with spatial and semantic components and apply the ideas to a partial implementation in the object-oriented DBMS O_2.

A *multi-resolution model* of a geographic space affords representations at a variety of levels of detail as well as providing a structure in which these representations are located. In such models, generalization operators are required in order that transitions between different locations in the structure can be made. Puppo and Dettori [PD95] provide a formal model of some of the topological and metric aspects of multi-resolution using abstract cell complexes and homotopy theory - both topics within algebraic topology. These ideas are further developed by Bertolotto [Ber98], who proposes a definition of a base set of transformations, from which set a significant class of generalization operators can be obtained. This body of work provides one of the motivations for the current work, in that the earlier work does not seek to make a distinction between *selection* of features, where certain features of the phenomena are chosen and others omitted, and *amalgamation* of features, where certain features originally considered distinguishable are made indistinguishable. In current multi-resolution models, selection and amalgamation are not distinguished, yet they are conceptually quite distinct.

We use the term *simplification* for a generalization which can be described as a selection followed by an amalgamation. We are aware that 'simplification' does refer to a very specific operation in the literature on cartographic generalization, and that we are using the word in a more general sense here. However, this word has been used by Puppo and Dettori [PD95, p161] in a way that fits very closely with the present paper. Puppo and Dettori define 'simplification mappings' which are certain mappings between cell complexes. A particular simplification mapping $F : \Gamma \to \Gamma'$ provides a reason why the cell complex Γ' is a simplified version of Γ. This leads to a category [BW95], where the objects are cell complexes and the morphisms are the simplification mappings. In our work we have a category where the objects are graphs, and the morphisms are simplifications in our sense.

Further work on the amalgamation properties of multi-resolution data models is discussed by Worboys [Wor98b], where a lattice of resolution is constructed and properties of entities represented at differing degrees of granularity considered. This theme is pursued further in [Wor98a] by showing that the resolution lattice can be applied to both geometric and semantic resolutions. Stell and Worboys [SW98] develop the formal properties of the resolution lattice, showing how each resolution in the lattice gives rise to a space of spatial data representations all with respect to that resolution, and the totality of spatial data representations being stratified by the resolution lattice. Generalization operators and their inverses can be considered as transitions between layers in the stratified spatial data space. Stell has also recently provided [Ste99] an analysis of different notions of granularity for graphs.

2.2 Handling Graphs in Databases

In this paper, the techniques developed with regard to selection and amalgamation operators are applied as transitions between resolutions in a multi-resolution data model in the particular context of graphs. This is done for three reasons:

1. Graphs provide an intermediate level between non-spatial data and full planar spatial data, and are sufficiently rich to illustrate the application of the approach.
2. Graphs have many applications in spatial information systems, for example road and rail networks, and cable and other utility networks.

3. The dual graph of an areal partition of the plane (where nodes of the dual graph are associated with areas in the partition, and nodes are connected by an edge if and only if the areas are adjacent in the partition) is an important indicator of the topological relationships between the areas in the partition.

The database community has put some effort into considering how generic database technology can be used to provide functionality for handling graph data structures. Mannino and Shapiro [MS90] survey work on extending the relational database model to incorporate functionality for handling graphs, including extensions to query languages for graph traversal. Güting [Güt92], presented an approach that extended the relational data model with data types for planar-embedded graphs. Güting's 1992 paper was followed by a series of papers in which he and colleagues developed the theme of incorporating graph handling capabilities in database systems [Güt94,EG94,BG95]. Erwig and Schneider [ES97] pose the question of the meaning of vagueness with reference to a graph. Stell and Worboys [SW97] have discussed the algebraic structure of the set of subgraphs of a graph.

3 Selection and Amalgamation

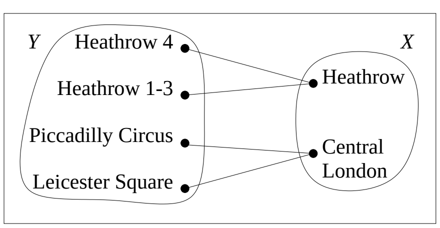
A major motivation of this work is to clarify the distinction between selection and amalgamation generalization operations. In this section we explore the foundations of the concept *"less detailed than"*, based on the notions of selection and amalgamation. At the most abstract level, there are two ways in which data represented by a structure X can be less detailed than data represented by a structure Y.

selection: X can be derived from Y by selecting certain features, and possibly leaving out others.

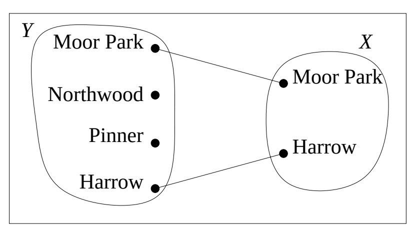
amalgamation: X can be derived from Y by amalgamating some features of Y so that some distinct things in Y are regarded as indistinguishable and become just one thing in X.

3.1 Amalgamation and Selection for Sets

The two operations are illustrated in the case of sets X and Y, the simplest formal structures, by the following concrete examples.



Amalgamation



Selection

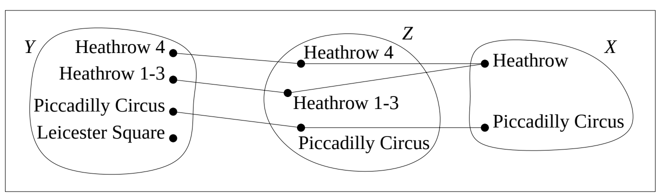
In the amalgamation example, the set Y consists of four stations on the London Underground. For some application it may be inappropriate to distinguish between the two individual stations at Heathrow Airport. Similarly, the stations Piccadilly Circus and Leicester Square are physically close together, and at a lower level of detail, the distinction between them may not be important. By avoiding the distinctions between these pairs of stations, we arrive at the set X as a less detailed representation of the data in Y.

The example of selection is also derived from actual data about the London Underground. Here again Y is a set representing four individual stations which is represented at a lower level of detail by a set X containing only two elements. However, in this case the operation performed on Y to produce X is quite different. The stations present in X are selected from those in Y because of their relative importance. Northwood and Pinner are minor stations, and many trains which do stop at Moor Park and Harrow do not stop at the two smaller stations.

When X and Y are sets it is straightforward to formalize the notions of amalgamation and selection. If the relationship of X to Y is one of selection, then there is an injective (or one-to-one) function from X to Y. If the relationship is one of amalgamation, then there is a surjective (or onto) function from Y to X.

3.2 Combining Amalgamation and Selection for Sets

The above examples deal with two ways in which X may be a less detailed representation of Y. In more complicated examples the relationship need not be solely one of amalgamation or selection. In general, a loss of detail relationship between X and Y will involve both selection and amalgamation. This entails a set Z which is obtained from Y by selection, and which is amalgamated to produce X. A simple example appears in the the following diagram.



A pair consisting of a selection followed by an amalgamation will be called a *simplification* from Y to X. Formally, a simplification from a set Y to a set X consists of three things: a set Z, an injective function from Z to Y (the selection part) and a surjective function from Z to X (the amalgamation part). Alternatively we can describe a simplification from Y to X as a partial surjective function from Y to X.

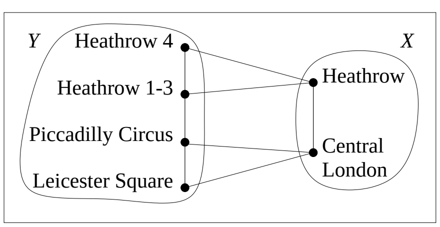
It might appear that by defining a simplification to consist of a selection followed by an amalgamation, we are being unnecessarily restrictive. It is natural to ask whether this definition of simplification excludes an amalgamation followed by a selection, or a sequence of the form $[s_1, a_1, s_2, a_2, \cdots, s_n, a_n]$ where each a_i is an amalgamation, and each s_i is a selection. In fact, provided we are dealing with simplifications of graphs, or of sets, every sequence of the above form can be expressed as a single selection followed by a single amalgamation. The justification for this lies in the fact that simplifications can be composed. For simplifications of sets, this is discussed in the following paragraph. For simplifications of graphs, composition is illustrated by an example in section 4, and defined formally in section 5.4. It is worth noting that a single selection on its own is still a simplification. This is because it can be expressed as a selection followed by the trivial amalgamation in which no distinct entities are amalgamated. Similarly, a single amalgamation on its own is a simplification, since it is equal to the trivial selection, which selects everything, followed by the amalgamation.

A simplification from Y to X gives a way of modelling a *reason why* X is less detailed than Y. As the earlier examples showed, X can be a simplification of Y for many different reasons, thus it is necessary to keep track of the specific amalgamations and selections involved. It is also important to be able to compose simplifications. If σ_1 is a simplification from Y to X, and σ_2 a simplification from X to W, we need to be able to construct a simplification $\sigma_1;\sigma_2$ from Y to W which represents σ_1 followed by σ_2. The usual notion of composition for partial functions provides the appropriate construction in the current context.

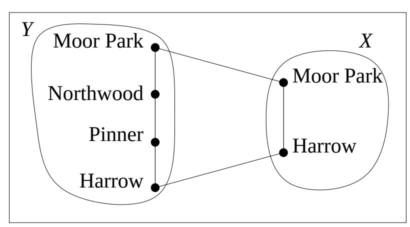
3.3 Amalgamation and Selection for Graphs

Simplifications between sets are a useful way of illustrating the concepts of selection and amalgamation, but to handle more complex kinds of data we need more elaborate structures than sets. In this section simple examples of amalgamation and selection for graphs are introduced. Further examples appear in the detailed case study in section 4.

The following two examples develop the previous treatment of amalgamation and selection for sets by adding edges between the elements of the sets to represent how the stations are joined by railway lines.



Amalgamation



Selection

In the amalgamation example, the two stations at Heathrow airport collapse into a single entity, as before, but note that the edge between them in Y is not present in X. This disappearance of an edge can be understood in terms of amalgamations of paths. Roughly speaking, a path is a sequence of zero, one, or more edges in which each edge in the sequence shares one end with the next edge in the sequence and one end with the previous edge in the sequence. However, because we are using undirected edges, a more careful formal treatment is needed, and appears in section 5 below. In the concrete example being discussed here, four paths in Y become amalgamated into a single edge in X. The four paths distinguished in Y which are amalgamated in X are as follows.

Heathrow 4	Heathrow 1-3	Piccadilly Circus	Leicester Square
	Heathrow 1-3	Piccadilly Circus	Leicester Square
Heathrow 4	Heathrow 1-3	Piccadilly Circus	
	Heathrow 1-3	Piccadilly Circus	

In the selection example, the edge in the graph X is not selected from the edges present in Y, but is selected from the paths in Y. The use of paths in both the amalgamation and selection operations is an important feature of our work. Formally we treat amalgamations and selections as particular kinds of morphisms between graphs. These morphisms are mappings taking nodes to nodes, but which may map edges to paths, and not merely to edges.

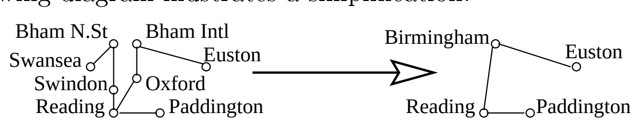
A significant distinction between our work and that of both Puppo and Dettori [PD95] and Bertolotto [Ber98] is illustrated in the selection example. As a graph is a particular kind of 1-dimensional abstract cell complex, the technique of using continuous mappings between abstract cell complexes to model simplifications, which these authors use, can be applied to graphs. However, this technique would force us to use a mapping sending the two intermediate stations as well as the three edges in the graph Y to the single edge in the graph X. Conceptually this act of amalgamating Northwood and Pinner stations with three railway lines appears inappropriate if we want to model the simple idea that our graph X is obtained from Y by *omitting* certain features altogether. While continuous mappings between abstract cell complexes may be suitable for some kinds of simplification, they do not seem adequate to model the concept of selection.

These two examples of amalgamation and selection for graphs illustrate only a few of the features of our approach. As with sets, general loss of detail relationships between graphs involve both amalgamation and selection. Examples showing how amalgamation and selection are combined into simplifications for graphs, and how simplifications of graphs are composed appear in the detailed case study in section 4 below.

4 Case Study

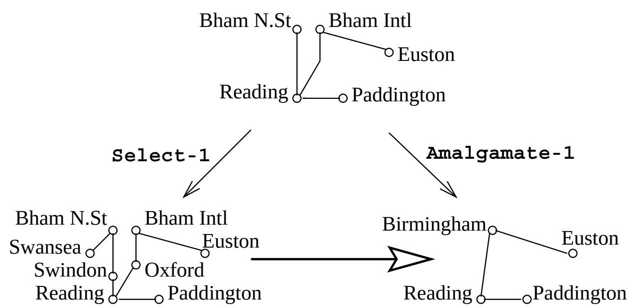
In this section we present a detailed case study showing how our concepts of amalgamation and selection can be combined to yield a notion of simplification for graphs. The case study is drawn from genuine examples of the railway network in Britain.

The following diagram illustrates a simplification:



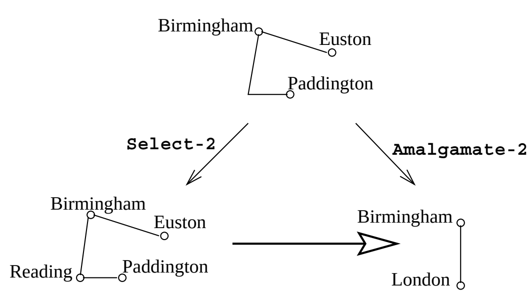
At the most detailed level, two stations in Birmingham are shown: Birmingham New Street and Birmingham International. These are amalgamated at the lower level. The line from Swansea to Birmingham New Street, as well as Swansea station itself are omitted at the lower level, as are the two stations intermediate between Reading and Birmingham. The route from Reading to Birmingham New Street is amalgamated with that from Reading to Birmingham International.

The simplification is made up of a selection and an amalgamation as in the following diagram:

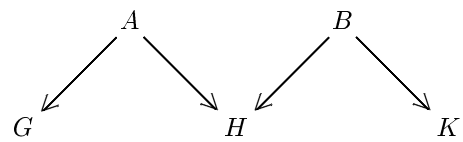


Note that a selection from a graph G need not be obtained by selecting some of the nodes and some of the edges from G. We allow a selection to take paths and not just edges from G. This technique allows selections to omit intermediate stations, such as Swindon, without being forced to omit railway lines passing through such stations. This means that we can model the fact that a line joins Reading to Birmingham New Street, even though no single edge represents this at the highest level. This use of paths is an important aspect of our work, formally it amounts to working with morphisms between graphs which take edges to paths, and is detailed in section 5 below.

The graph which appeared as the end result of the above simplification can be simplified further as in the following diagram. Here the two stations in London: Euston and Paddington have been amalgamated, as have the two routes from London to Birmingham. Reading station has also been omitted.

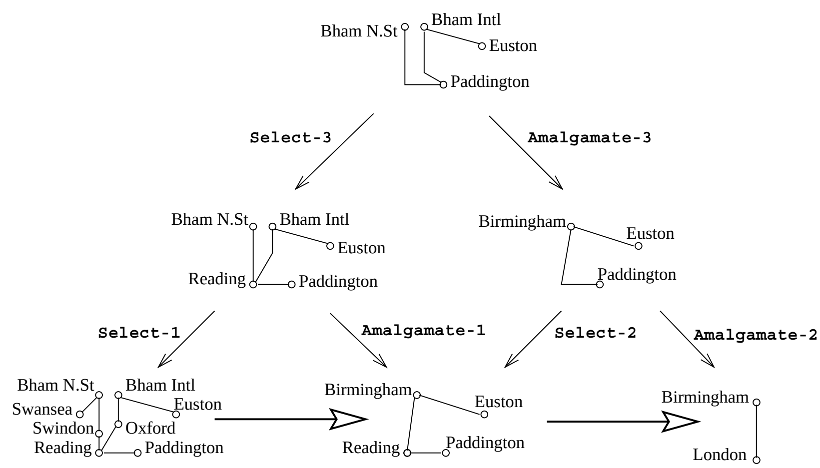


We now have two successive simplifications involving five graphs altogether as follows:

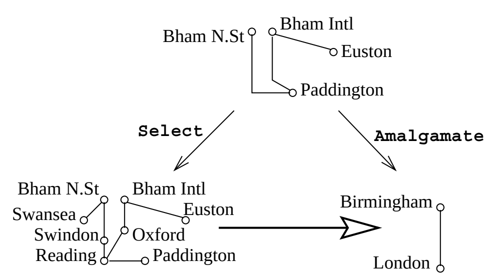


and we wish to express this as a single simplification.

The basic idea is to note that from A to B we have an amalgamation (to H) followed by a selection (from H). It is possible to interchange these, so that we can obtain B from A by first performing a selection (`Select-3`) and then an amalgamation (`Amalgamate-3`). A formal description of this construction is provided in section 5.4 below, but here we provide a diagram showing the result for our specific example.



By composing `Select-3` with `Select-1`, and by composing `Amalgamate-3` with `Amalgamate-2` we obtain a single simplification:



This case study has demonstrated the main points of our technique, but has not included the full details necessary to produce an implementation. A full account of the technical details is included in section 5 below.

5 Technical Details

To model structures and simplifications between them, including composition of successive simplifications, it is appropriate to use the mathematical structure known as a category [BW95]. Categories have already been used in the context of multi-resolution spatial data models by Bertolotto [Ber98]. However, unlike Bertolotto, our treatment is based on the category **Graph***, which is described in section 5.2 below. In order to present this material, some basic facts about graphs are needed first.

5.1 The Category Graph

The graphs used in this paper are undirected, are permitted to have loops, and may have multiple edges between the same pair of nodes. Much of the work in the paper can be carried out for directed graphs, but some aspects become slightly more complicated, while other aspects are easier to deal with. Limitations on space prevent us from giving details of both the undirected and directed cases.

The set of all subsets having either one or two elements, of a set, N, is denoted by $\mathbb{P}_2 N = \{\,\{x, y\} \mid x, y \in N\}$. A graph is then described formally as a pair of sets E and N (of edges and nodes respectively), together with an incidence function $i : E \to \mathbb{P}_2 N$.

A graph morphism $f : \langle E_1, i_1, N_1\rangle \to \langle E_2, i_2, N_2\rangle$ is a pair of functions $f_E : E_1 \to E_2$ and $f_N : N_1 \to N_2$, such that if the ends of edge $e \in E_1$ are x and y, then the ends of $f_E e \in E_2$ are $f_N x$ and $f_N y$. These morphisms take edges to edges in a way which preserves the incidence function. Given a graph morphism, f, the two functions f_E and f_N are referred to as the edge part and the node part of the morphism respectively. The category **Graph** has graphs as objects, and graph morphisms as its morphisms.

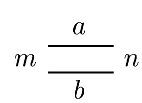
5.2 The Category Graph*

To define simplifications of graphs, we need another category which has the same objects as **Graph**, but more general morphisms. Given a graph G define the graph G^* to have the same nodes as G, and as edges, the set of all paths in G. A path in G can be described by a sequence, of nodes and edges of the form

$$[x_0, e_1, x_1, e_2, \ldots, e_\ell, x_\ell] \qquad (1)$$

where edge e_k has ends x_{k-1} and x_k. The case of $\ell = 0$ is allowed, and gives zero length paths which are loops on each node. Two different sequences represent the same path iff each is the reverse of the other. The ends of the path are x_0 and x_ℓ.

Sometimes it is appropriate to write a path simply as $[e_1, e_2, \ldots, e_\ell]$, but in general this can be ambiguous. For example, consider the following graph with two nodes, m and n, and two edges, a and b.



The paths $[m, a, n, b, m]$ and $[n, a, m, b, n]$ are quite distinct, and simply using the sequence of edges $[a, b]$ in this context would be ambiguous.

The $*$ construction is applicable not only to graphs, but also to morphisms. Given any graph morphism $f : G \to H$ we can construct a graph morphism $f^* : G^* \to H^*$. The morphism f^* has the same effect as f on nodes, and takes the edge (1) above to $[fx_0, fe_1, fx_1, fe_2, \ldots, fe_\ell, fx_\ell]$.

The graph G can always be embedded in G^* by the **Graph** morphism $\eta_G : G \to G^*$ which takes each node to itself, and an edge e to the path $[e]$ of length 1. Repeating the $*$ construction leads to a graph $(G^*)^*$. This has the same nodes as G and G^*, but the edges are paths of paths of G, which have the form

$$[x_0, [x_0, \sigma_1, x_1], x_1, [x_1, \sigma_2, x_2], \ldots, x_{n-1}, [x_{n-1}, \sigma_n, x_n], x_n] \qquad (2)$$

where each σ_k is a sequence of edges and nodes of G, of the form

$$e_1, y_1, e_2, \ldots, e_{m-1}, y_{m-1}, e_m.$$

It is possible to reduce, or 'flatten', an edge in $(G^*)^*$ to one in G^* by mapping the edge (2) above to $[x_0, \sigma_1, x_1, \sigma_2, x_2, \ldots, x_{n-1}, \sigma_n, x_n]$. This assignment gives the edge part of a morphism $\mathsf{flat}_G : (G^*)^* \to G^*$ which is the identity on nodes.

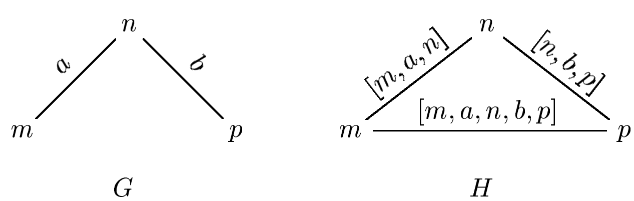
The $*$ construction allows us to define the category **Graph***. This has the same objects as **Graph**, but morphisms from G to H in **Graph***are ordinary graph morphisms from G to H^*. Given morphisms $f : G \to H$ and $g : H \to K$ in **Graph***, their composition is given by

$$G \xrightarrow{\ f\ } H^* \xrightarrow{\ g^*\ } (K^*)^* \xrightarrow{\ \mathsf{flat}_K\ } K^*$$

5.3 Selection and Amalgamation for Graphs

Definition 1 *A* selection *from a graph G is a subgraph A of G^* such that for each path π in G, there is at most one path ψ in A where flattening ψ yields π.*

The following example should help to clarify this definition.



In the above diagram we have a graph, G, and a graph H. The graph H is a subgraph of G^*, but is not a selection from G. This is because the two paths in H: $[m, [m, a, n], n, [n, b, p], p]$ and $[m, [m, a, n, b, p], p]$ both flatten to the same path $[m, a, n, b, p]$ in G.

If A is a selection from G and C is a selection from A, one might hope that C would be a selection from G. However, this does not happen, as can be seen from a simple example. Let G be the graph with three edges and four nodes $m \overset{a}{\text{———}} n \overset{b}{\text{———}} p \overset{c}{\text{———}} q$, and let A be the selection from G: $m \overset{[m,a,n,b,p]}{\text{—————}} p \overset{[p,c,q]}{\text{———}} q$. The graph C: $m \overset{[m,[m,a,n,b,p],p,[p,c,q],q]}{\text{——————————}} q$ is a selection from A, but not a selection from G, since it is a selection from G^*. This failure of selections to compose is overcome by noting that applying flat_G to C yields a selection from G which is isomorphic to C itself. For our specific example, $\mathsf{flat}_G\, C$ is the graph $m \overset{[m,a,n,b,p,c,q]}{\text{—————}} q$

Definition 2 *An* amalgamation *for a graph G is a* **Graph** *morphism $\alpha : G \to H^*$, such that the node part of the morphism, α_N, is surjective, and for every edge e of H there is some edge e' of G for which $\alpha_E e' = [e]$, where α_E is the edge part of the morphism.*

Definition 3 *A* simplification *from a graph G to a graph H, is a pair (A, α) where A is a selection from G, and $\alpha : A \to H^*$ is an amalgamation of A.*

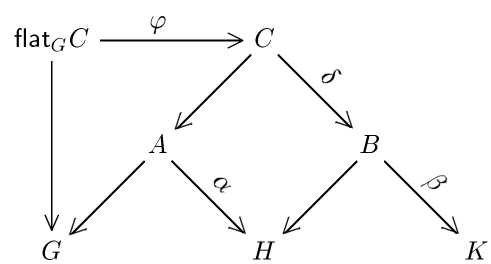
5.4 A Construction for Composing Simplifications

If we have two successive simplifications of graphs $G \xrightarrow{(A,\alpha)} H \xrightarrow{(B,\beta)} K$ we need to be able to compose them to give a simplification $G \xrightarrow{(A,\alpha);(B,\beta)} K$. This is done by first constructing the graph C, which is the largest subgraph, X, of A^* such that $\alpha^* X = B$. By restricting α^* to C, we obtain a **Graph** morphism $\delta : C \to B^*$, so that B is an amalgamation of C.

Now C is a subgraph of A^*, and hence of $(G^*)^*$, whereas for a simplification from G to K, we need a subgraph of G^*. By applying the construction for selections of selections above, we get $\mathsf{flat}_G C$ as a selection from G, and an isomorphism $\varphi : \mathsf{flat}_G C \to C$. Finally we get the definition of the composite simplification

$$(A, \alpha); (B, \beta) = (\mathsf{flat}_G C, \varphi; \delta; \beta),$$

where $\varphi; \delta; \beta$ denotes the composite of these three morphisms in **Graph***. The overall picture is seen in this diagram in the category **Graph***:



With this method of composing simplifications, we have a category where the objects are graphs, and the morphisms are simplifications.

6 Conclusions and Further Work

The work described in this paper has concerned explication of and distinction between the generalization operations that select and amalgamate data objects in a data model of a spatial phenomenon. In previous work, the functions of these two conceptually quite distinct operations have been conflated. We have termed such operations simplifications, and have provided a formal treatment of simplification in the case where the underlying data structure is a graph. In particular, we have shown that the appropriate formal home for these structures is the category **Graph***, and in that context it is possible to provide a construction for composition of simplification operations. We justified concentration on graph data structures because they were simple enough to show up clearly the main structural features of our approach, while at the same time being useful in spatial information handling with a history of treatment by SIS researchers.

Our approach is limited in several ways. Firstly, the simplification operators considered are in no way claimed to be a complete set of generalization operations, and further work is required to incorporate into this framework a richer collection of generalization operations. Secondly, the graph data structures are one-dimensional. The next step in the work is to consider simplification operators in fully two-dimensional data structures, in particular 2-complexes. The technical details for this case are more difficult. For example, the notion of a path of edges in a graph must be generalized to a gluing together of faces in a 2-complex. Work in this direction will be reported in a future publication.

There is one particular construction in which graph data structures have immediate application to fully two-dimensional data, and that is to areal decompositions of the plane. The *dual graph* of such a decomposition is a graph, where areas in the decomposition are nodes of the graph and two nodes are connected by an edge in the graph if the corresponding areas are adjacent in the decomposition. Some forms of generalization of an areal decomposition can be viewed as simplifications of its dual graph. For example, merging of two adjacent areas in an areal decomposition is equivalent in the dual graph to amalgamation of two nodes and elimination of the edge between them. One direction for further work using dual graphs would be to develop the boundary sensitive approach to qualitative location [BS98,Ste99] in a multi-resolution context. A detailed exploration of generalizations of areal decompositions of the plane in terms of simplification of the dual graph will be reported in later publications.

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