M. Egenhofer and J. Sharma
Topological Relations Between Regions in R^2 and Z^2
Advances in Spatial Databases--Third International Symposium on Large Spatial Databases, SSD `93, Singapore
D. Abel and B.C. Ooi (eds.), Lecture Notes in Computer Science, Vol. 692, Springer-Verlag, pp. 316-336, June 1993.

Topological Relations Between Regions in $\mathbb{R}^2$ and $\mathbb{Z}^2$*

Max J. Egenhofer and Jayant Sharma

National Center for Geographic Information and Analysis
and
Department of Surveying Engineering
Department of Computer Science
University of Maine
Orono, ME 04469-5711, U.S.A.
max@mecan1.maine.edu
jayant@thrush.umesve.maine.edu

Abstract. Users of geographic databases that integrate spatial data represented in vector and raster models, should not perceive the differences among the data models in which data are represented, nor should they be forced to apply different concepts depending on the model in which spatial data are represented. A crucial aspect of spatial query languages for such integrated systems is the need mechanisms to process queries about spatial relations in a consistent fashion. This paper compares topological relations between spatial objects represented in a continuous (vector) space of $\mathbb{R}^2$ and a discrete (raster) space of $\mathbb{Z}^2$. It applies the 9-intersection, a frequently used formalism for topological spatial relations between objects represented in a vector data model, to describe topological relations for bounded objects represented in a raster data model. We found that the set of all possible topological relations between regions in $\mathbb{R}^2$ is a subset of the topological relations that can be realized between two bounded, extended objects in $\mathbb{Z}^2$. At a theoretical level, the results contribute toward a better understanding of the differences in the topology of continuous and discrete space. The particular lesson learnt here is that topology in $\mathbb{R}^2$ is based on coincidence, whereas in $\mathbb{Z}^2$ it is based on coincidence and neighborhood. The relevant differences between the raster and the vector model are that an object's boundary in $\mathbb{Z}^2$ has an extent, while it has none in $\mathbb{R}^2$; and in the finite space of $\mathbb{Z}^2$ there are points between which one cannot insert another one, while in the infinite space of $\mathbb{R}^2$ between any two points there exists another one.

1 Introduction

Spatial relations are significant ingredients of query languages for geographic information systems (GISs), where they are used to describe constraints among spatial objects to be retrieved or updated. Relations among spatial objects are less well understood than the commonly used relations among integers or strings [18] and

---

* This work was partially supported by grants from Intergraph Corporation. Jayant Sharma was supported by a University Graduate Research Assistantship (UGRA) from the University of Maine. Additional support from NSF for the NCGIA under grant No. SES 88-10917 is gratefully acknowledged.

attempts to formalize them have been rare. Recently, progress has been made on selected topics such as cardinal directions between point objects [14, 20] and some combinations of relations such as topological and direction relations [2, 25, 28].

Topological relations have been studied extensively in the past and there exists a comprehensive formalism [7, 8]. This model has become popular in the GIS community and has been used extensively for a variety of purposes. Svensson and Zhexue [37] incorporated it into a spatial-analysis query language. De Hoop and van Oosterom [27] designed a topological query language around these operators. Herring [26] discussed possible extensions of the model to cover topological relations between lines in $\mathbb{R}^2$. Egenhofer and Al-Taha [12] used the model to investigate various aspects of temporal changes of topological relations. Pigot [32] and Hazelton *et al.* [23] showed how the model applies to 3-dimensional objects; Hadzilacos and Tryfona [22] and Clementini *et al.* [4] showed how the model for regions behaves when applied to model topological relations between regions, lines, and points; and Egenhofer and Herring [9] extended the principle of boundary and interior intersections to include intersections with the exterior as well as a generic model for topological relations involving *n*-dimensional objects embedded in higher-dimensional spaces. Parts of these extensions are now being tested with human subjects to identify how closely the formalism models human cognition [30]. The sound model also enabled a number of advanced theoretical studies such as the formal derivation of the composition table for this set of relations [6] and comprehensive reasoning systems to detect inconsistencies in topological descriptions [11, 35]. A derivative of the method has also been successfully implemented in a commercial GIS [26].

This paper extends the scope of this model by investigating its usefulness for modeling topological relations among bounded objects embedded in the discrete space $\mathbb{Z}^2$. Its goal is to answer such questions as, “Are the topological relations between bounded objects in $\mathbb{R}^2$ and $\mathbb{Z}^2$ the same?” and if they are different then, “What are their differences?” By basing the investigations of raster relations on the same model as vector relations, we expect to make formal comparisons in the model, rather than giving intuitive interpretations.

Such an approach is a significant contribution towards designing spatial information systems that integrate what is usually called a “vector” and “raster” representation of spatial objects [16, 21], which is a topic that has gotten considerable attention through the discussions of integrating remotely sensed data with GIS data in vector format [13]. Much of this discussion has been at the level of data structures. The actual problems faced as one attempts to merge a data model for continuous space with a model of discrete space, are deeper in nature. To date, there exist only a few approaches that try to combine the two views at a conceptual level [31]. In order to come up with an integrated model it is definitely necessary to formalize the differences at the conceptual level, rather than the level of particular data structures or implementations. This paper contributes towards such an integrated data model for geographic databases, as it identifies topological properties that are common or different between the two views. As such, the results of this paper will help in gaining a better understanding of the differences between raster and vector space.

The remainder of this paper is structured as follows. Section 2 briefly summarizes the basic concepts of the 9-intersection applied for topological relations between regions in $\mathbb{R}^2$. Following an introduction of a data model for spatial regions in $\mathbb{Z}^2$ (Section 3), we derive formally the existing topological relations based on the 9-intersection (Section 4). Section 5 compares the two sets of topological relations by analyzing the differences of $\mathbb{R}^2$ and $\mathbb{Z}^2$ that caused them, and by giving a cognitive interpretation in terms of their conceptual neighborhoods. Conclusions in Section 6 show how the results also apply to “coarse” spatial reasoning.

2 The 9-Intersection as a Model for Topological Relations in $\mathbb{R}^2$

The usual concepts of point-set topology with open and closed sets are assumed [1, 36]. The interior of a set A, denoted by A°, is the union of all open sets in A. The closure of A, denoted by $\overline{A}$, is the intersection of all closed sets of A. The exterior of A with respect to the embedding space $\mathbb{R}^2$, denoted by A^-, is the set of all points of $\mathbb{R}^2$ not contained in A. The boundary of A, denoted by ∂A, is the intersection of the closure of A and the closure of the exterior of A. The spatial objects of concern are called spatial regions. A *spatial region* is defined as 2-dimensional point-set that is homeomorphic to a 2-disk, i.e., each of the three object parts of a region—its interior, boundary, and exterior—is non-empty and connected.

For two regions A and B, the binary topological relation between them is characterized by comparing A's boundary (∂A), interior (A°), and exterior (A^-) with B's boundary (∂B), interior (B°), and exterior (B^-). These six object parts are combined such that they form nine intersections that represent the topological relation between the two regions. They are:

- the boundary-boundary intersection, denoted by $\partial A \cap \partial B$,
- the boundary-interior intersection, denoted by $\partial A \cap B^\circ$,
- the boundary-exterior intersection, denoted by $\partial A \cap B^-$,
- the interior-boundary intersection, denoted by $A^\circ \cap \partial B$,
- the interior-interior intersection, denoted by $A^\circ \cap B^\circ$,
- the interior-exterior intersection, denoted by $A^\circ \cap B^-$,
- the exterior-boundary intersection, denoted by $A^- \cap \partial B$,
- the exterior-interior intersection, denoted by $A^- \cap B^\circ$, and
- the exterior-exterior intersection denoted by $A^- \cap B^-$.

The topological relation between regions A and B, is concisely represented as a 3×3 matrix, called the *9-intersection.*

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \partial A \cap \partial B & \partial A \cap \partial B & \partial A \cap B^- \\ A^\circ \cap \partial B & A^\circ \cap \partial B & A^\circ \cap B^- \\ A^- \cap \partial B & A^- \cap \partial B & A^- \cap B^- \end{pmatrix}$$

Topological relations are characterized by *topological invariants* of the 9-intersections, i.e., properties that are preserved under topological transformations. The *content* of the nine intersections was identified as the simplest and most general topological invariant [7], though others may be useful as well such as the components of an intersection and their dimensions [17]. The content invariant characterizes each of the nine intersections by the value empty ($\varnothing$) or non-empty ($\neg\varnothing$). With the empty/non-empty distinction of the nine intersections, one can distinguish $2^9 = 512$ different topological relations. Exactly one of these topological relations holds true between any two regions, because the nine empty/non-empty intersections describe a set of relations that are mutually exclusive and provide a complete coverage—the three object parts boundary, interior, and exterior cover the entire universe as a complete partition of space; and the contents of their intersections are such that any set is either empty or non-empty. The actual number of realizable relations depends on the dimension of the space with respect to the dimension of the objects—there are more topological relations if objects are embedded in higher-dimensional space—and on topological properties of the objects embedded in that space. For example, the boundary of a (non-cyclic) line—the set of its start and end points—is a separation, whereas the boundary of a region without holes is connected, and the difference in these topological properties influences what topological relations can be realized.

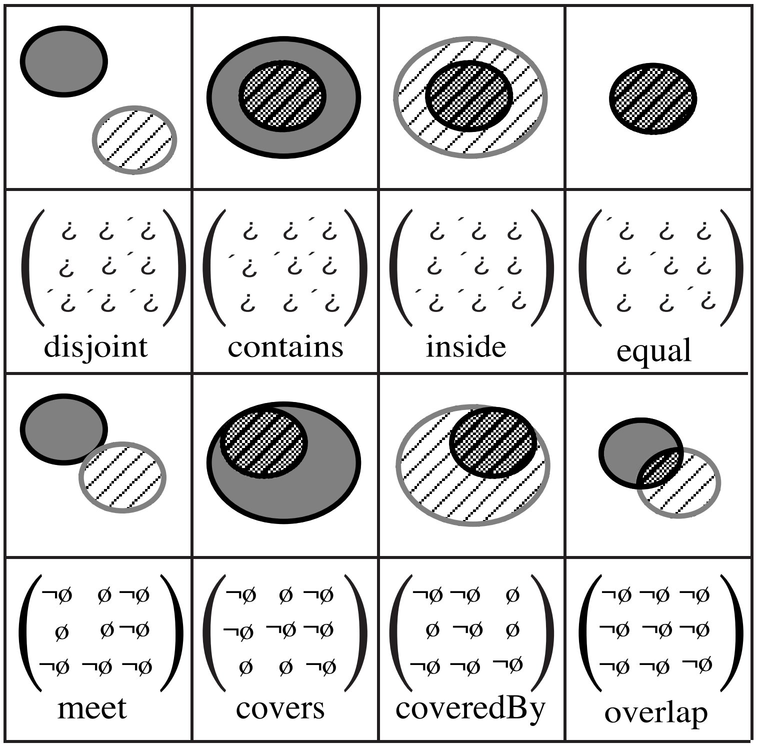


Fig. 1. The eight relations between two regions in $\mathbb{R}^2$.

From the 512 possible relations, only eight can be realized if the objects are spatial regions in $\mathbb{R}^2$. We call these eight relations *disjoint*, *meet*, *equal*, *inside*, *contains*, *covers*, *coveredBy*, and *overlap* (Fig. 1). Recently, these results were verified using an independent method [33] that is based on an interval logic about connections rather than set theory with intersections.

3 Spatial Data Model for Objects in $\mathbb{Z}^2$

The present investigations concentrate on topological relations in raster space. Raster space, or the digital plane $\mathbb{Z} \times \mathbb{Z}$, is defined as a rectangular array of points or pixels. Each point is addressed by a pair of integer valued coordinates (x, y). We will briefly review the concepts most relevant for the subsequent discussions. More extensive and detailed treatments of digital topology can be found in [24, 34].

Given a point in the plane, the neighboring points can be classified as 4-neighbors or 8-neighbors. The 4-neighbors of a point *P* are the vertically and horizontally adjacent points. Along with the diagonally adjacent points, they form the 8-neighbors. Fig. 2 gives an example of the 4- and 8-neighbors of a point.

Fig. 2. A point with its 4- and 8-neighbors.

A finite proper subset of $\mathbb{Z}^2$ is called an extended spatial object. Any two points *P* and *Q* that belong to an extended object *R* are connected if there exists a connected path between them, i.e., a sequence of adjacent points, all in *R*, that starts at *P* and ends at *Q*. If all adjacent points in the path are 4-neighbors then *P* and *Q* are connected by a 4-path. *R* is 4-connected if for any pair of points *P*, *Q* in *R* there exists a 4-path of finite length between them. The corresponding definition holds for 8-connectedness.

The objects of concern are *raster regions*, i.e., extended objects that are bounded. The boundary of a raster region is a simple closed curve *C*, which divides the background into two components and every point on *C* is adjacent to both these components; therefore, the boundary of a raster region is a Jordan Curve [38], which separates the region into an interior and an exterior (Fig. 3).

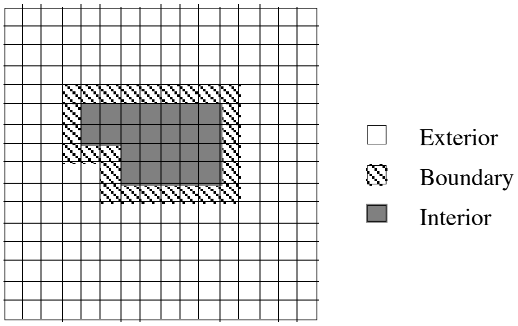


Fig. 3. The boundary, interior, and exterior of a region.

Raster regions have the following properties:

- Boundary and interior are non-empty (Fig. 4a). (The non-emptiness of the exterior is already guaranteed by R being a proper subset of $\mathbb{Z}^2$.)
- The boundary is 4-connected such that each boundary point has exactly two 4-neighbors (Fig. 4b and c). This implies that the boundary separates the interior from the exterior such that there are no two points, one in the interior and one in the exterior, that are 4- or 8-eight connected.
- The exterior of a raster region is 4-connected, which excludes any interior holes as well as regions that touch at their own boundaries (Fig. 4d).
- The interior of a raster region is 4-connected such that each interior point has at least three 8-neighbors (Fig. 4e).

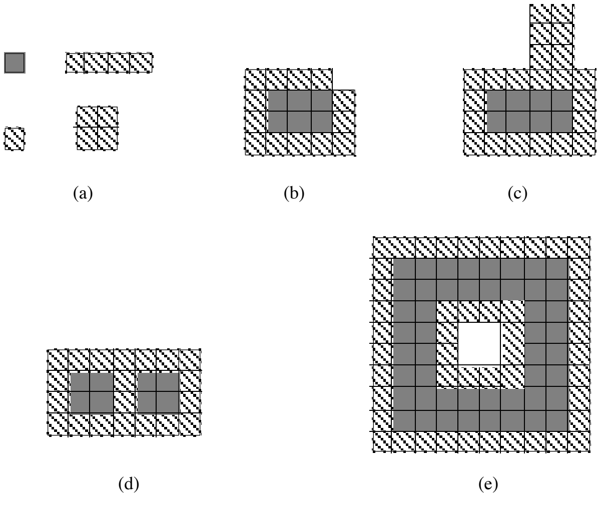


Fig. 4. Disallowed raster regions: (a) empty interior or boundary; (b) the boundary is not 4-connected; (c) a boundary point with more than two 4-neighbors; (d) disconnected interiors; and (e) disconnected exteriors.

M. Egenhofer and J. Sharma
Topological Relations Between Regions in R^2 and Z^2
Advances in Spatial Databases--Third International Symposium on Large Spatial Databases, SSD `93, Singapore
D. Abel and B.C. Ooi (eds.), Lecture Notes in Computer Science, Vol. 692, Springer-Verlag, pp. 316-336, June 1993.

Furthermore, we exclude some degenerate cases—very small and very large objects—by limiting the raster regions to have the following properties:

- Each point in the interior of a raster region has at least three 8-neighbors.
- The union of no pair of raster regions can occupy $\mathbb{Z}^2$ completely.

4 Realizable 9-Intersections in $\mathbb{Z}^2$

This section examines the binary topological relations between raster regions in $\mathbb{Z}^2$. As the underlying model, we use the 9-intersection whose empty/non-empty invariant gives rise to 2^9, i.e., 512 different relations. Here, we will determine which 9-intersections represent feasible topological relations between raster regions embedded in $\mathbb{Z}^2$. The result will enable us to compare raster-region relations with vector-region relations.

The identification of 9-intersections for which topological relations exist in $\mathbb{Z}^2$ is a four-step process:

(1) Describing the conditions that must hold among interiors, boundaries, and exteriors in order to guarantee consistency within the data model of raster regions. The conditions are based upon the mathematical properties of the objects and the underlying space, such as the Jordan-Curve-Theorem, and restrictions on objects, e.g., a region must have a non-empty interior and a non-empty exterior.

(2) Formalization of all conditions such that they can be integrated and compared. Rather than finding all constraints that describe the *valid* configurations among interiors, boundaries, and exteriors, we pursue the opposite, that is, we collect a set of constraints that describe *invalid* configurations and infer as candidates for valid configurations those that do not violate any constraint. This is achieved by translating all conditions among boundaries, interiors, and exteriors into *consistency violations* and expressing them as templates of 9-intersections for non-existing relations.

(3) Determining the set of 9-intersections that represent existing relations. This is done by applying the templates for non-existing conditions and removing matching patterns from the set of all 512 possible 9-intersections. Note that the conditions are not mutually exclusive and therefore, a particular 9-intersection may match more than one template.

(4) Verifying the existence of corresponding topological relations for this set of 9-intersections by finding geometric configurations for them.

Subsequently, we present a set of conditions that lead to the set of binary topological relations between two regions in a raster space.

M. Egenhofer and J. Sharma
Topological Relations Between Regions in R^2 and $\mathbb{Z}^2$
Advances in Spatial Databases--Third International Symposium on Large Spatial Databases, SSD `93, Singapore
D. Abel and B.C. Ooi (eds.), Lecture Notes in Computer Science, Vol. 692, Springer-Verlag, pp. 316-336, June 1993.

4.1 Consistency Constraints among Interiors, Boundaries, and Exteriors

Raster regions are 2-dimensional and embedded in $\mathbb{Z}^2$, therefore, any part of a region—its interior, boundary, or exterior—constrains the location of the other two parts. Thus, one can infer from the fact that two parts of a raster region coincide that the other corresponding parts must coincide as well:

Condition 1a: If the interiors of two raster regions coincide, then the regions' exteriors and boundaries must coincide as well.

Condition 1b: If the boundaries of two raster regions coincide, then the regions' interiors and exteriors coincide equal as well.

Condition 1c: If the exteriors of two raster regions coincide, then the regions' interiors and boundaries must coincide as well.

The boundary of a raster region forms a Jordan Curve, separating the interior from the exterior such that any connected path from the interior to the exterior has to intersect with the boundary. This leads to the following six conditions for connected object parts:

Condition 2a: If A's interior intersects with B's interior *and* exterior then it must also intersect with B's boundary and vice versa.

Condition 2b: If A's boundary intersects B's interior *and* exterior then it must also intersect with B's boundary and vice versa.

Condition 2c: If A's exterior intersects with B's interior *and* exterior then it must also intersect with B's boundary and vice versa.

Condition 2d: If A's boundary intersects with the boundary, interior, *and* exterior of B then A's interior must intersect with B's boundary, and vice versa.

Condition 2e: If A's boundary is a subset of B's closure, then A's interior must be a subset of B's interior, and vice versa.

Condition 2f: If both interiors are disjoint, then each interior must intersect with the other regions' exterior.

Since the regions are small with respect to the embedding space, we can claim that no two regions will cover the entire universe.

Condition 3: The exteriors of two regions must intersect with each other.

4.2 Translating Consistency Constraints into 9-Intersection Templates

In order to compare the conditions among interiors, boundaries, and exteriors it is necessary to represent them in a unifying model. The 9-intersection serves as the basis for such a model. Since the content of some intersections may be irrelevant for some consistent configurations, we introduce a “wild card” (_) to denote intersections whose content may be either empty or non-empty [9]. A *9-intersection template* is a pattern of empty, non-empty, or arbitrary intersections which represent constraints among interiors, boundaries, and exteriors. The following example illustrates how a constraint among boundary and interior can be formalized as a 9-intersection template. If A's boundary is disjoint from B's interior then $\partial A \cap B^{\circ}$ must be empty, while the values of the other eight intersections do not matter:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \_ & \varnothing & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix}$$

A constraint about a valid configuration can be transformed into a constraint about an invalid configuration, because empty/non-empty 9-intersections are a closed system providing complete coverage. For example, "the interior must intersect with at least one part of the other object" is equivalent to the expression, "it is invalid if all three intersections of the interior with the boundary, interior, and exterior of the other object are empty." In terms of the 9-intersection this means:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \_ & \_ & \_ \\ \varnothing & \varnothing & \varnothing \\ \_ & \_ & \_ \end{pmatrix}$$

Such translations may be applied for all conditions, transforming a consistency constraint that must hold for all valid configurations into a consistency constraint that cannot hold for any valid configuration. Appendix A shows the complete set of 9-intersection templates for non-existing relations.

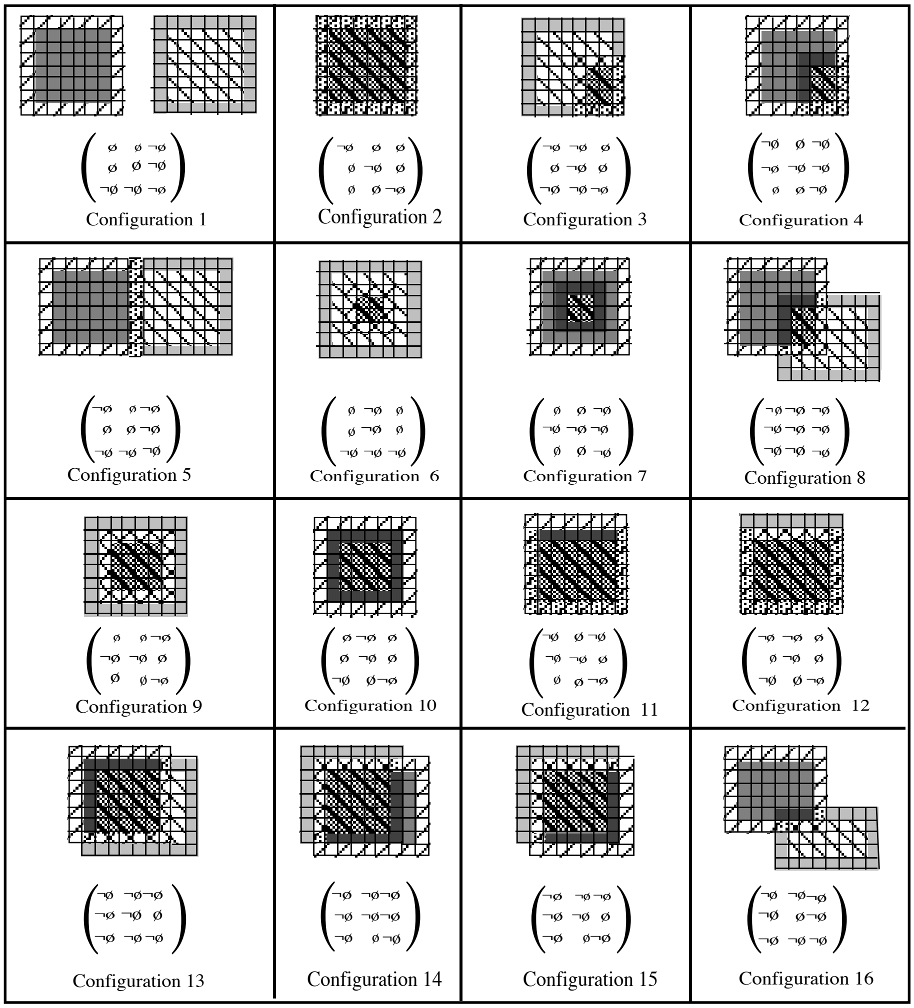
4.3 Realization of Region Relations in $\mathbb{Z}^2$

The set of 9-intersection templates for non-existing relations serves two purposes: (1) to find the smallest set of constraints and (2) to find the 9-intersections that are candidates for relations that may be realized in $\mathbb{Z}^2$.

Multiple conditions may be correlated because the same non-existing relation, described by two patterns of 9-intersections, can be a member of different conditions. By determining a minimum set-cover [3] of all 9-intersection templates for non-existing relations one can eliminate such redundant constraints. We found that condition (1b) is implied by the other constraints and, therefore, can be left out. The minimum set cover shows also that conditions (1a) and (1c) can be simplified into what is shown as conditions (1a') and (1c') in the Appendix.

By successively eliminating the 9-intersections for invalid configurations from the set of all 512 possible empty/non-empty intersections, one determines the set of 9-intersections that are expected to be realized in $\mathbb{Z}^2$. We programmed this stepwise elimination. It leaves a set of sixteen 9-intersections. For each of these we found a geometric interpretation, shown in Fig. 5 together with their corresponding

9-intersections. In lieu of assigning them names, we have chosen to number them 1…16. We can, therefore, conclude that these sixteen topological relations are the complete set of region relations that can be realized with empty/non-empty 9-intersections in $\mathbb{Z}^2$.



$$\begin{pmatrix} \emptyset & \emptyset & \neg\emptyset \\ \emptyset & \emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 1

$$\begin{pmatrix} \neg\emptyset & \emptyset & \emptyset \\ \emptyset & \neg\emptyset & \emptyset \\ \emptyset & \emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 2

$$\begin{pmatrix} \neg\emptyset & \neg\emptyset & \emptyset \\ \emptyset & \neg\emptyset & \emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 3

$$\begin{pmatrix} \neg\emptyset & \emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \\ \emptyset & \emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 4

$$\begin{pmatrix} \neg\emptyset & \emptyset & \neg\emptyset \\ \emptyset & \emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 5

$$\begin{pmatrix} \emptyset & \neg\emptyset & \emptyset \\ \emptyset & \neg\emptyset & \emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 6

$$\begin{pmatrix} \emptyset & \emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \\ \emptyset & \emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 7

$$\begin{pmatrix} \neg\emptyset & \neg\emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 8

$$\begin{pmatrix} \emptyset & \emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \emptyset \\ \emptyset & \emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 9

$$\begin{pmatrix} \emptyset & \neg\emptyset & \emptyset \\ \emptyset & \neg\emptyset & \emptyset \\ \neg\emptyset & \emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 10

$$\begin{pmatrix} \neg\emptyset & \emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \emptyset \\ \emptyset & \emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 11

$$\begin{pmatrix} \neg\emptyset & \neg\emptyset & \emptyset \\ \emptyset & \neg\emptyset & \emptyset \\ \neg\emptyset & \emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 12

$$\begin{pmatrix} \neg\emptyset & \neg\emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 13

$$\begin{pmatrix} \neg\emptyset & \neg\emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \\ \neg\emptyset & \emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 14

$$\begin{pmatrix} \neg\emptyset & \neg\emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \emptyset \\ \neg\emptyset & \emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 15

$$\begin{pmatrix} \neg\emptyset & \neg\emptyset & \neg\emptyset \\ \neg\emptyset & \emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 16

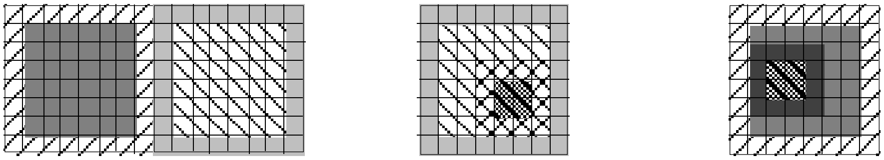
Fig. 5. The sixteen relations between two raster regions.

4.4 Beyond the Content Invariant

The 9-intersection may be analyzed using criteria other than the content of the intersections to describe more detailed topological relations [8, 26]. Examples of such additional criteria applied to regions in $\mathbb{R}^2$ are the dimension, the type of boundary-

boundary intersection (touching from outside, touching from inside, or crossing), and the sequence of the components (i.e., the largest connected subset) of boundary-boundary intersections [17].

For relations between raster regions, it is most important to determine whether two boundaries are neighbors or not. This criterion is special for relations in $\mathbb{Z}^2$, as it represents a particular property of a discrete space (see Section 5.1). Two boundaries, ∂A and ∂B, are neighbors if $\partial A \cap \partial B$ is empty and there exist at least two points, $a \in \partial A$ and $b \in \partial B$, such that a is a 4- or 8-neighbor of b. In general, such refinements are possible for all 9-intersections with empty boundary-boundary intersections. Exceptions are the 9-intersections for which a different boundary neighborhood would change their empty/non-empty specifications. This is the case with configurations 9 and 10, which have empty boundary-boundary intersections, but all boundary points must be 4- or 8-neighbors. If they were not neighbors, then their 9-intersections were the same as configurations 6 and 7, respectively. Therefore, only for configurations 1 and 6—and its converse configuration 7—can one find more detailed relations if the neighborhood criterion of boundary-boundary intersections is considered. These detailed relations will be called 1a, 6a, and 7a (Fig. 6).



$$\begin{pmatrix} \emptyset & \emptyset & \neg\emptyset \\ \emptyset & \emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix} \qquad \begin{pmatrix} \emptyset & \neg\emptyset & \emptyset \\ \emptyset & \neg\emptyset & \emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix} \qquad \begin{pmatrix} \emptyset & \emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \\ \emptyset & \emptyset & \neg\emptyset \end{pmatrix}$$

Configuration 1a Configuration 6a Configuration 7a

Fig. 6. Detailed raster region relations for 9-intersections with empty boundary-boundary intersections.

5 Comparing Existing Topological Relations in $\mathbb{R}^2$ and $\mathbb{Z}^2$

The comparison of topological relations in $\mathbb{R}^2$ and $\mathbb{Z}^2$ can be discussed at three different levels: (1) the pure comparison of 9-intersections that can be realized in the two spaces; (2) an analysis of the conditions that lead to the different sets; and (3) a cognitive assessment of the two sets of relations through their conceptual neighborhoods. The first discussion will identify particular topological properties that result from the 9-intersection. The comparison of the different constraints will contribute to a better understanding of the differences between continuous and discrete space. Finally, from the cognitive analysis we expect to learn what kinds of relations may be necessary in a spatial query language of an integrated raster-vector system.

5.1 Additional 9-Intersections and Conditions for Relations in $\mathbb{R}^2$

Comparing the sixteen raster relations with the eight vector relations, one finds that all 9-intersections of the vector relations are a subset of the raster relations; therefore, the set of constraints in $\mathbb{Z}^2$ may be obtained as an extension of the set of constraints in $\mathbb{R}^2$. The following two constraints among interiors, boundaries, and exteriors of regions in $\mathbb{R}^2$ reduce the set of sixteen raster relations to the set of eight relations that can be realized in $\mathbb{R}^2$. They hold only for relations between vector regions, not for raster regions, and have to be applied *in addition* to conditions (1-3).

Condition 4: If A's interior intersects with B's boundary then it must also intersect with B's exterior, and vice-versa.

This condition eliminates seven of the sixteen raster relations (Configurations 9–15).

Condition 5: If both interiors are disjoint then A's boundary cannot intersect with B's interior, and vice-versa.

This condition eliminates the last raster relation that cannot be realized in $\mathbb{R}^2$ (Configuration 16).

These two additional conditions can be interpreted in several ways. First, they reflect the difference between (discrete) raster space and (continuous) vector space: Two distinct pixels in a raster may be adjacent to each other such that they have no other pixel in between, while in vector space any two distinct points always have another point between them. A second interpretation of the two rules is that the topological properties of a vector space can be achieved by shrinking the (extended) boundary of a raster region so that the boundary "disappears" to the mere border line between an interior and an exterior pixel. The latter interpretation leads also to the conclusion that the topology of a raster space is obtained by making the boundaries of the vector regions "broad." All these interpretations are only valid for the limited set of raster regions considered, i.e., that each pixel in the region's interior must have at least three 8-neighbors.

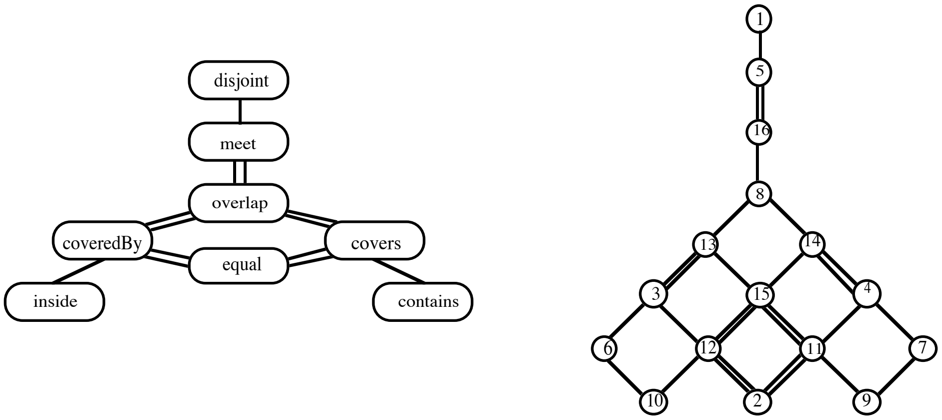
5.2 Conceptual Neighborhoods

While the differences between the two sets of relations are relevant for the processing of spatial queries, they are also significant for the use of these spatial concepts in spatial query languages. The questions arises, "should the query language of a raster-based GIS have a different set of topological relations than a vector-based GIS?" From the perspective of a system designer, the previous results may suggest that it is necessary to do so. On the other hand, users should not be made aware of the particular data model of a GIS when they query about spatial relations. Since we found that the set of region relations in $\mathbb{R}^2$ is a subset of the ones in $\mathbb{Z}^2$, the relations in

$\mathbb{R}^2$ may be considered as the “integral set.” Our goal here is to find the mappings from raster relations onto vector relations that would be most reasonable. Certainly these mappings should conform with cognitive measures, i.e., the “most similar” relations should be mapped onto a single relation.

For this goal, we are comparing the *conceptual neighborhoods* [19] of both sets of relations. Conceptual neighborhoods identify those relations that are close to each other and yield information about cognitive aspects of the relations, such as their behavior under specific deformations. They have been successfully applied to relations between 1-dimensional intervals [19], cardinal directions in 2-D [20], and topological relations between regions in $\mathbb{R}^2$ [12]. Two topological relations are *conceptual neighbors* if the transition from one relation to another will be “smooth,” such that no other relation is between the two relations when applying a gradual change. Such gradual changes may be deformations to one of the two regions involved, such as scaling, rotation, and translation, that do not change the topology of the region.

For topological relations, the *topology distance* has been proposed as a measure to determine conceptual neighbors [12]. The topology distance is the number of differences in corresponding values of two 9-intersections. Informally the topology distance between two 9-intersections is the smallest number of “bits” that must be flipped to convert one 9-intersection into the other. The topology distance between a relation and itself is 0, and it is between 1 and 9 for any other pair of topological relations. For example, the topology distance from *disjoint* to *meet* is 1. The shorter the topology distance between two relations, the smaller is their conceptual difference. Pairs of raster relations with topology distance 1 and 2 provide a connected graph in which each node corresponds to a topological relation and each edge, linking two relations, denotes that these relations are conceptual neighbors.



(a)

(b)

Fig. 7. The conceptual neighbors of topological relations in (a) vector and (b) raster space. (Single lines indicate neighbors with topology distance 1,

while double lines stand for neighbors with topology distance 2 or more.)

The three additional raster relations obtained from considering 4- and 8-neighbors of non-empty boundary-boundary intersections (Section 4.4) can be added into the conceptual neighborhoods: Configuration 1a is located between 1 and 5; 6a is equally close to 3 and 10 from 6; and 7a is equally close to 4 and 9 from 7 (Fig. 8). This addition to the conceptual neighborhood graph adds edges whose topology distance is 0 (from 1 to 1a; 6 to 6a; and 7 to 7a).

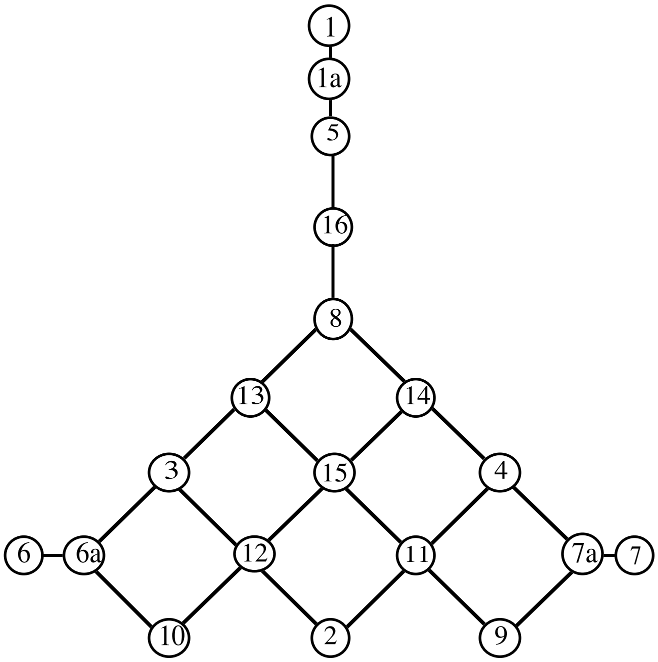
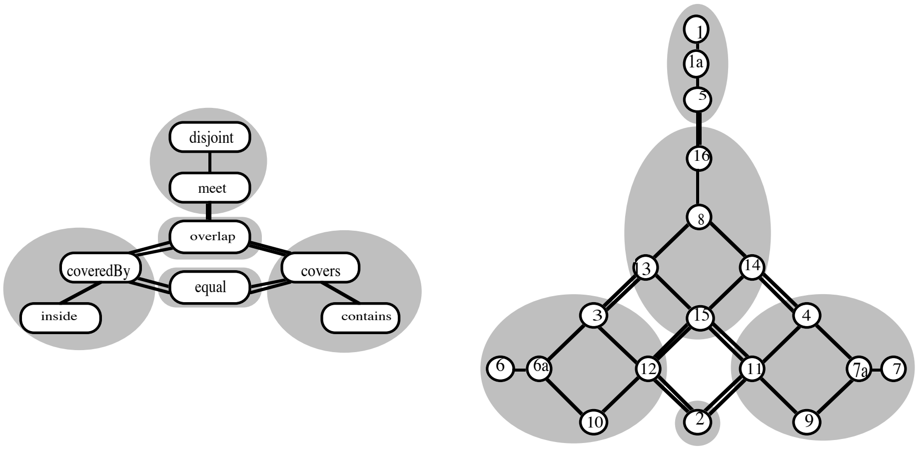


Fig. 8. The extended raster relation graph, including topological relations obtained from empty boundary-boundary intersections being 4- or 8-neighbors.

A *conceptual neighborhood cluster* groups together all relations that are transitively within 1 topology distance unit. The comparison of the two conceptual neighborhoods reveals that both sets of topological relations have the same neighborhood clusters and that corresponding clusters have the same links (of topology distance 2) with other clusters (Fig. 9).



(a) (b)

Fig. 9. The conceptual neighborhood clusters for (a) vector relations and (b) raster relations.

With the help of these five common neighborhood clusters, we find the following mappings from raster relations onto vector relations:

- In each space, there is one (topological) equivalence relation, therefore, the raster configuration 2 maps onto the vector configuration *equal*.
- The four “new” raster relations 13, 14, 15, and 16, which are grouped around configuration 8, map onto the vector relation *overlap*.
- Configurations 3, 6, 6a, 10, and 12 map onto *inside/coveredBy*. The analysis of how the neighborhood clusters are interrelated reveals that 3 and 12 correspond to *coveredBy*, while 6 maps onto *inside*. This leaves the question where 6a and 10 belong. One can make a case for either grouping—putting them with *inside* or with *coveredBy*. The corresponding mappings can be done for the cluster with *contains/covers*.
- Configurations 1, 1a, and 5 map onto the cluster *disjoint/meet*. Within this cluster, 1 corresponds to disjoint—they are both at the end of the graph. More difficult is the assessment of where 1a belongs, as it can be grouped with either *disjoint* or *meet*. It might even be argued that 5 belongs into the *overlap* cluster and 1a is the only raster relation that corresponds to *meet*.

The ambiguities in some of these mappings may indicate that both situations may occur, depending on the spatial concept used. For example, if individual objects, such as buildings, are represented in a raster model, then two buildings may *meet* according to configuration 1a; whereas two objects in a partition of space (e.g., land parcels) would *meet* according to configuration 5.

6 Conclusions

We presented a comparison of binary topological relations between bounded objects in $\mathbb{R}^2$ and $\mathbb{Z}^2$, based on the empty/non-empty intersections of interiors, boundaries, and exteriors. The results are:

- All eight topological relations that can be realized between two vector regions can be realized between two raster regions as well.
- There are eight more topological relations between two bounded raster regions than there are between two bounded vector regions.
- Another three topological relations may be identified in $\mathbb{Z}^2$ if one considers whether non-empty boundary-boundary intersections are neighbors.
- Through the analysis of the conceptual neighborhoods of both sets of relations, we found mappings for the additional relations onto the common eight relations.
- The crucial difference between the two models is that the boundary of a raster region has an extent (it is one pixel wide), while the boundary of a vector region has none.

The results are germane to the integration of raster and vector GISs [13]. The approach chosen is significantly different from previous integration attempts [31] as it identify common properties at the level of spatial data models, rather than comparing implementation aspects of data structures [10, 15].

The results are also significant for reasoning in geographic (large-scale) space, which has to account for imprecise information. A particular category of geographic objects are those whose boundaries are not well-defined such as the Rocky Mountains, the Gulf of Mexico, or the Great Lakes region. Despite a lack of precision, humans can effectively reason about such objects—Aspen is in the Rockies; Hurricane Andrew crossed the Gulf of Mexico; or a storm front is approaching the Great Lakes. Current spatial data models for geographic information systems do not account for the representation of and reasoning about such “imprecise” objects. There have been attempts, however, to describe the boundaries through “fuzzy” lines [5, 29]. The results may be considered as a new approach to “coarse” topological reasoning, where objects are represented as “big blocks” whose boundaries have an extent. Such a representation corresponds to a raster model in which each object’s boundary is made up of cells or pixels, rather than being the (invisible) separation between the interior and the exterior of the object.

Appendix A: Consistency Constraints for Non-Existing Topological Relations in $\mathbb{Z}^2$

Condition 1a:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \_ & \varnothing & \_ \\ \varnothing & \neg\varnothing & \varnothing \\ \_ & \varnothing & \varnothing \end{pmatrix} \vee \begin{pmatrix} \varnothing & \varnothing & \_ \\ \varnothing & \neg\varnothing & \varnothing \\ \_ & \varnothing & \_ \end{pmatrix} \vee \begin{pmatrix} \_ & \varnothing & \neg\varnothing \\ \varnothing & \neg\varnothing & \varnothing \\ \_ & \varnothing & \_ \end{pmatrix} \vee \begin{pmatrix} \_ & \varnothing & \_ \\ \varnothing & \neg\varnothing & \varnothing \\ \neg\varnothing & \varnothing & \_ \end{pmatrix}$$

Condition 1a':

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \varnothing & \varnothing & \_ \\ \varnothing & \neg\varnothing & \varnothing \\ \_ & \varnothing & \_ \end{pmatrix} \vee \begin{pmatrix} \_ & \varnothing & \neg\varnothing \\ \varnothing & \neg\varnothing & \varnothing \\ \_ & \varnothing & \_ \end{pmatrix} \vee \begin{pmatrix} \_ & \varnothing & \_ \\ \varnothing & \neg\varnothing & \varnothing \\ \neg\varnothing & \varnothing & \_ \end{pmatrix}$$

Condition 1b:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \neg\varnothing & \varnothing & \varnothing \\ \varnothing & \_ & \_ \\ \varnothing & \_ & \varnothing \end{pmatrix} \vee \begin{pmatrix} \neg\varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \_ \\ \varnothing & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \neg\varnothing & \varnothing & \varnothing \\ \varnothing & \_ & \neg\varnothing \\ \varnothing & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \neg\varnothing & \varnothing & \varnothing \\ \varnothing & \_ & \_ \\ \varnothing & \varnothing & \_ \end{pmatrix}$$

Condition 1c:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \_ & \_ & \varnothing \\ \_ & \varnothing & \varnothing \\ \varnothing & \varnothing & \neg\varnothing \end{pmatrix} \vee \begin{pmatrix} \varnothing & \_ & \varnothing \\ \_ & \_ & \varnothing \\ \varnothing & \varnothing & \neg\varnothing \end{pmatrix} \vee \begin{pmatrix} \_ & \neg\varnothing & \varnothing \\ \_ & \_ & \varnothing \\ \varnothing & \varnothing & \neg\varnothing \end{pmatrix} \vee \begin{pmatrix} \_ & \_ & \varnothing \\ \neg\varnothing & \_ & \varnothing \\ \varnothing & \varnothing & \neg\varnothing \end{pmatrix}$$

Condition 1c':

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \_ & \neg\varnothing & \varnothing \\ \_ & \_ & \varnothing \\ \varnothing & \varnothing & \neg\varnothing \end{pmatrix} \vee \begin{pmatrix} \_ & \_ & \varnothing \\ \neg\varnothing & \_ & \varnothing \\ \varnothing & \varnothing & \neg\varnothing \end{pmatrix}$$

Condition 2a:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \_ & \_ & \_ \\ \varnothing & \neg\varnothing & \neg\varnothing \\ \_ & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \_ & \varnothing & \_ \\ \_ & \neg\varnothing & \_ \\ \_ & \neg\varnothing & \_ \end{pmatrix}$$

Condition 2b:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \varnothing & \neg\varnothing & \neg\varnothing \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \varnothing & \_ & \_ \\ \neg\varnothing & \_ & \_ \\ \neg\varnothing & \_ & \_ \end{pmatrix}$$

Condition 2c:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix} \vee \begin{pmatrix} \_ & \_ & \emptyset \\ \_ & \_ & \neg\emptyset \\ \_ & \_ & \neg\emptyset \end{pmatrix}$$

Condition 2d:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \neg\emptyset & \neg\emptyset & \neg\emptyset \\ \emptyset & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \neg\emptyset & \emptyset & \_ \\ \neg\emptyset & \_ & \_ \\ \neg\emptyset & \_ & \_ \end{pmatrix}$$

Condition 2e:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \_ & \neg\emptyset & \emptyset \\ \neg\emptyset & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \_ & \neg\emptyset & \_ \\ \neg\emptyset & \_ & \_ \\ \emptyset & \_ & \_ \end{pmatrix}$$

Condition 2f:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \_ & \_ & \_ \\ \_ & \emptyset & \emptyset \\ \_ & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \_ & \_ & \_ \\ \_ & \emptyset & \_ \\ \_ & \emptyset & \_ \end{pmatrix}$$

Condition 3:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \emptyset \end{pmatrix}$$

Condition 4:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \_ & \_ & \_ \\ \neg\emptyset & \_ & \emptyset \\ \_ & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \_ & \neg\emptyset & \_ \\ \_ & \_ & \_ \\ \_ & \emptyset & \_ \end{pmatrix}$$

Condition 5:

$$\mathbf{R}\,(A, B) = \begin{pmatrix} \_ & \_ & \_ \\ \neg\emptyset & \emptyset & \_ \\ \_ & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \_ & \neg\emptyset & \_ \\ \_ & \emptyset & \_ \\ \_ & \_ & \_ \end{pmatrix}$$

References

1. P. Alexandroff (1961) *Elementary Concepts of Topology*. Dover Publications, Inc., New York, NY.

2. S. K. Chang, Q. Y. Shi, and C. W. Yan (1987) Iconic Indexing by 2-D Strings. *IEEE Transactions on Pattern Analysis and Machine Intelligence* PAMI-9(6): 413-428.

3. V. Chávtal (1979) A Greedy Heuristic for the Set-Covering Problem. *Mathematics of Operations Research* 4(3): 233-235.

4. E. Clementini, P. di Felice, and P. van Oosterom (1993) A Small Set of Formal Topological Relationships Suitable for End-User Interaction. in: D. Abel and B. Ooi (Eds.), *Third International Symposium on Large Spatial Databases, SSD '93,* Singapore. *Lecture Notes in Computer Science.* Springer-Verlag, New York, NY.

5. S. Dutta (1991) Topological Constraints: A Representational Framework for Approximate Spatial and Temporal Reasoning. in: O. Günther and H.-J. Schek (Eds.), *Advances in Spatial Databases—Second Symposium, SSD '91*, Zurich, Switzerland. *Lecture Notes in Computer Science* 525, pp. 161-180, Springer-Verlag, New York, NY.

6. M. Egenhofer (1991) Reasoning about Binary Topological Relations. in: O. Günther and H.-J. Schek (Eds.), *Advances in Spatial Databases—Second Symposium, SSD '91*, Zurich, Switzerland. *Lecture Notes in Computer Science* 525, pp. 143-160, Springer-Verlag, New York, NY.

7. M. Egenhofer and R. Franzosa (1991) Point-Set Topological Spatial Relations. *International Journal of Geographical Information Systems* 5(2): 161-174.

8. M. Egenhofer and J. Herring (1990) A Mathematical Framework for the Definition of Topological Relationships. *Fourth International Symposium on Spatial Data Handling*, Zurich, Switzerland, pp. 803-813.

9. M. Egenhofer and J. Herring (1991) *Categorizing Binary Topological Relationships Between Regions, Lines, and Points in Geographic Databases*. Technical Report, Department of Surveying Engineering, University of Maine, Orono, ME (submitted for publication).

10. M. Egenhofer and J. Herring (1991) High-Level Spatial Data Structures for GIS. in: D. Maguire, M. Goodchild, and D. Rhind (Eds.), *Geographical Information Systems, Volume 1: Principles.* pp. 227-237, Longman, London.

11. M. Egenhofer and J. Sharma (1992) Topological Consistency. *Fifth International Symposium on Spatial Data Handling*, Charleston, SC, pp. 335-343.

12. M. J. Egenhofer and K. K. Al-Taha (1992) Reasoning About Gradual Changes of Topological Relationships. in: A. U. Frank, I. Campari, and U. Formentini (Eds.), *Theories and Models of Spatio-Temporal Reasoning in Geographic*

Space, Pisa, Italy. *Lecture Notes in Computer Science* 639, pp. 196-219, Springer-Verlag, New York.

13. M. Ehlers, G. Edwards, and Y. Bedard (1989) Integration of Remote Sensing with Geographic Information Systems: A Necessary Evolution. *Photogrammetric Engineering & Remote Sensing*. 55(11): 1619-1627.

14. A. Frank (1992) Qualitative Spatial Reasoning about Distances and Directions in Geographic Space. *Journal of Visual Languages and Computing* 3(4): 343-371.

15. A. Frank (1992) Spatial Concepts, Geometric Data Models and Data Structures. *Computers and Geosciences* 18(4): 409-417.

16. A. Frank and D. Mark (1991) Language Issues for GIS. in: D. Maguire, M. Goodchild, and D. Rhind (Eds.), *Geographical Information Systems, Volume 1: Principles*. pp. 147-163, Longman, London.

17. R. Franzosa and M. Egenhofer (1992) Topological Spatial Relations Based on Components and Dimensions of Set Intersections. *SPIE's OE/Technology '92—Vision Geometry*, Boston, MA.

18. J. Freeman (1975) The Modelling of Spatial Relations. *Computer Graphics and Image Processing*. 4: 156-171.

19. C. Freksa (1992) Temporal Reasoning Based on Semi-Intervals. *Artificial Intelligence* 54: 199-227.

20. C. Freksa (1992) Using Orientation Information for Qualitative Spatial Reasoning. in: A. U. Frank, I. Campari, and U. Formentini (Eds.), *Theories and Models of Spatio-Temporal Reasoning in Geographic Space*, Pisa, Italy. *Lecture Notes in Computer Science* 639, pp. 162-178, Springer-Verlag, New York.

21. M. Goodchild (1992) Geographical Data Modeling. *Computers and Geosciences* 18(4): 401-408.

22. T. Hadzilacos and N. Tryfona (1992) A Model for Expressing Topological Integrity Constraints in Geographic Databases. in: A. U. Frank, I. Campari, and U. Formentini (Eds.), *Theories and Models of Spatio-Temporal Reasoning in Geographic Space*, Pisa, Italy. *Lecture Notes in Computer Science* 639, pp. 252-268, Springer-Verlag, New York.

23. N. W. Hazelton, L. Bennett, and J. Masel (1992) Topological Structures for 4-Dimensional Geographic Information Systems. *Computers, Environment, and Urban Systems* 16(3): 227-237.

24. G. Herman (1990) On Topology as Applied to Image Analysis. *Computer Vision, Graphics, and Image Processing* 52: 409-415.

25. D. Hernández (1991) Relative Representation of Spatial Knowledge: The 2-D Case. in: D. Mark and A. Frank (Eds.), *Cognitive and Linguistic Aspects of Geographic Space*. pp. 373-385, Kluwer Academic Publishers, Dordrecht.

26. J. Herring (1991) The Mathematical Modeling of Spatial and Non-Spatial Information in Geographic Information Systems. in: D. Mark and A. Frank

(Eds.), *Cognitive and Linguistic Aspects of Geographic Space*. pp. 313-350, Kluwer Academic Publishers, Dordrecht.

27. S. de Hoop and P. van Oosterom (1992) Storage and Manipulation of Topology in Postgres. *Third European Conference on Geographical Information Systems, EGIS ‘92*. Munich, Germany, pp. 1324-1336.

28. S.-Y. Lee and F.-J. Hsu (1992) Spatial Reasoning and Similarity Retrieval of Images Using 2D C-String Knowledge Representation. *Pattern Recognition* 25(3): 305-318.

29. Y. Leung, M. Goodchild, and C.-C. Lin (1992) Visualization of Fuzzy Scenes and Probability Fields. *Fifth International Symposium on Spatial Data Handling*, Charleston, SC, pp. 480-490.

30. D. Mark and M. Egenhofer (1992) An Evaluation of the 9-Intersection for Region-Line Relations. *GIS/LIS ‘92*, San Jose, CA.

31. D. Peuquet (1988) Representations of Geographic Space: Toward a Conceptual Synthesis. *Annals of the Association of American Geographers* 78(3): 375-394.

32. S. Pigot (1991) Topological Models for 3D Spatial Information Systems. in: D. Mark and D. White (Eds.), *Autocarto 10*, Baltimore, MD, pp. 368-392.

33. D. Randell, Z. Cui, and A. Cohn (1992) A Spatial Logic Based on Regions and Connection. *Principles of Knowledge Representation and Reasoning, KR ‘92*, Cambridge, MA, pp. 165-176.

34. A. Rosenfeld (1979) Digital Topology. *American Mathematical Monthly* 86: 621-630.

35. T. Smith and K. Park (1992) Algebraic Approach to Spatial Reasoning. *International Journal of Geographical Information Systems* 6(3): 177-192.

36. E. Spanier (1966) *Algebraic Topology*. McGraw-Hill Book Company, New York, NY.

37. P. Svensson and H. Zhexue (1991) Geo-SAL: A Query Language for Spatial Data Analysis. in: O. Günther and H.-J. Schek (Ed.), *Advances in Spatial Databases—Second Symposium, SSD ‘91*, Zurich, Switzerland. *Lecture Notes in Computer Science* 525, pp. 119-140, Springer-Verlag, New York, NY.

38. A. Vince and C. Little (1989) Discrete Jordan Curve Theorems. *Journal of Combinatorial Theory Series B* 47: 251-261.