International Journal of Geographical Information Systems 7 (3): 215-229, 1993.

Modeling Spatial Relations and Operations with Partially Ordered Sets

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Abstract

Formal methods for the description of spatial relations can be based on mathematical theories of order. Subdivisions of land are represented as partially ordered sets (posets), a model that is general enough to answer spatial queries about inclusion and containment of spatial areas. After a brief introduction to the basic concepts of posets and lattices, their applications to modeling spatial relations and operations for spatial regions in terms of containment and overlay are presented. An interpretation is given for new geographic elements that are created by the completion from a poset to a lattice. It is shown that a novel approach to characterize certain topological relations based on a lattice of a simplicial complex is a model for spatial regions that combines both topological and order relations and allows spatial queries to be answered in a unified way.

1. Introduction

The modeling of spatial relations in geographic information systems (GISs) is based on fundamental mathematical theories. Over the past two decades it has been extensively investigated how to apply topology for modeling spatial relationships (Corbett 1979, White 1980, 1984, Frank and Kuhn 1986, Egenhofer and Franzosa 1991). The findings of these investigations have significantly contributed to the state-of-the-art data models used in modern GISs, such as Arc/Info (Morehouse 1985), TIGRIS (Herring 1987), System 9 (Charlwood *et al*. 1987) and Smallworld GIS (Newell *et al*. 1991), and the U.S. SDTS spatial data exchange standard (Rossmeissl and Rugg 1991). At the same time, only little attention has been paid to the application of complementary or alternative mathematical concepts to GIS and spatial analysis such as metric or order. Although Marvin White's (1984) landmark paper on the use of topology for cartography mentioned partially ordered sets and despite potential for interesting spatial applications of partially ordered sets, this topic has gained only little attention from GIS researchers. Saalfeld (1983a, 1983b, 1985), Meixler and Saalfeld (1985), Greasley (1988, 1990) and Kainz (1988, 1989, 1990) investigated some aspects of partial order relations in GIS. Nevertheless, to date there exists no comprehensive review of partially ordered sets and assessment of their usefulness in GIS applications. This paper attempts to fill this gap by providing an introduction to the mathematics of partially ordered sets and demonstrating their application to modeling spatial relationships in GISs.

The use of partially ordered sets is motivated here by an example to model spatial subdivisions of land. A spatial subdivision is commonly thought of as a hierarchy in which each area belongs to exactly one larger area. For example, most political subdivisions form a hierarchy—a country being divided into states, the states into counties, the counties into townships. Unfortunately, the model of a hierarchy is frequently insufficient to model spatial situations. For instance, a city may cross county boundaries. The following example is to demonstrate why users may not get

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answers to certain spatial queries if the spatial situation is modeled as a hierarchy. Consider two crop zones, A and B, and some fields a, b, c, d in these zones (Figure 1).

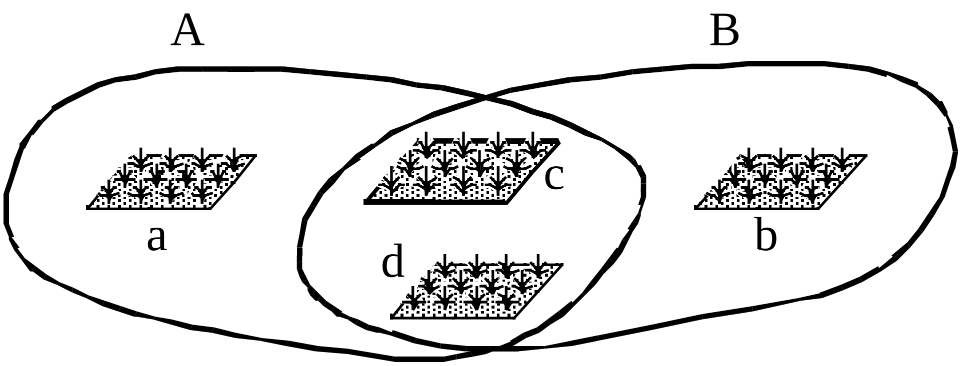

Figure 1: Spatial subdivision with fields and overlapping crop zones.

While a hierarchical structure is appropriate to answer queries about fields that are located in a single zone, it fails for those areas in which both A and B can be grown. Simple questions such as, "What can be grown in field c?" or "Where can I grow A and B?" cannot be answered, because field c would have to be either assigned to zone A or B, but not to both (Figure 2).

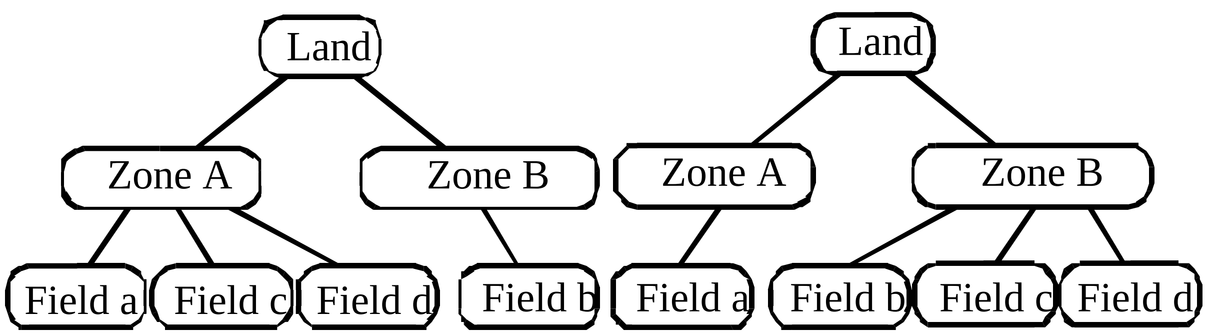

Figure 2: Representations of the subdivision in Figure 1 as hierarchies.

The mathematical structure of a partially ordered set overcomes many of the hierarchy's limitations and is a better quality model for such a form of geographic information than a hierarchical model. In fact, a hierarchy is a special type of partially ordered set where there is only one path from any element to the top.

The presentation of possible applications of partially ordered sets to GIS is built upon the basic theory of partially ordered sets and lattices. The concepts relevant to modeling spatial relations in GIS and pertinent to the upcoming discussions are briefly summarized in Section 2. Readers familiar with this mathematical background may skip this section and continue with Section 3, which discusses the meaning of certain order theoretic properties in terms of spatial subdivisions. Section 4 focuses on the possible applications of this theory to determine topological spatial relations such as containment and neighborhood operations. Conclusions in Section 5 compare the models of answering spatial queries either in the category of ordered sets or in the category of topological spaces, and outlines the needs for future research in this area.

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2. The Mathematics of Partially Ordered Sets

One of the basic structures upon which mathematical disciplines are built is *order*. A set is said to be (partially) ordered when an order relation is defined between its elements, making them comparable. The study of partially ordered sets and lattices is covered by an extensive amount of mathematical literature. Birkhoff's (1967) and Grätzer's (1978) books are considered to be classics. A recent work by Davey and Priestley (1990) provides an excellent introduction to the theory and gives many examples and applications in various disciplines. While some of these results are germane to modeling spatial relations, a great amount of this theoretical work appears to have little or no meaningful applications to geographic objects. To prevent the interested reader from laboring through the extensive volumes of mathematical literature, a brief summary of the most important concepts of partially ordered sets and lattices is given in this section. This section also introduces the notations used in the later sections.

2.1. *Posets*

Let P be a set. A *partial order* on P is a binary relation ≤ on P such that, for every $x, y, z \quad P$:

(i) $x \leq x$ (reflexive)
(ii) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetric)
(iii) if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitive)

A set P with a reflexive, antisymmetric and transitive relation (*order relation*) ≤ is called a *partially ordered set* (or *poset*). For every partially ordered set P we can find a new poset, the *dual* of P, by defining that $x \leq y$ is the dual if and only if $y \leq x$ in P. Any statement about a partially ordered set can be turned into a statement of its dual by replacing ≤ with ≥, and vice versa.

Each (finite) poset can be represented graphically by a *diagram*, also called the *Hasse diagram*. The construction of a diagram is based on the concept of a covering relation. "*A covers B*" (or "*B is covered by A*") in a poset P means that $B \leq A$ and there exists no $x \quad P$ that $B < x < A$, i.e., A is greater than B and there is no other element in between. The set of all elements that cover an element X is called the *cover* of X, denoted by X^-. Conversely, the set of all elements that are covered by X is called the *cocover* of X, written as X_-. A diagram of a poset P is drawn as a configuration of circles—representing the elements of P—and connecting lines—indicating the covering relation. If A covers B, the circle for element A is drawn above the circle for element B, and both are connected with a straight line (Figure 3).

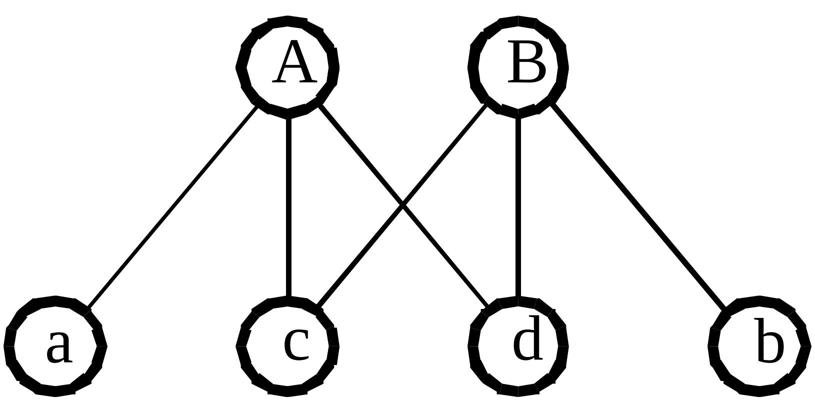

Figure 3: Poset representing the order in Figure 1.

For posets and subsets of ordered sets it is often desirable to have elements that are greater or less than all other elements. Let P be a poset and $S \quad P$. An element $a \quad S$ is the *greatest* (or *maximum*) *element* of S if $a \geq x$ for every $x \quad S$, denoted by $a = \max S$. The greatest element of P, if it exists, is called the *top element* of P. The *least* (or *minimum*) *element* of *S*, written as $\min S$, and the *bottom element* of P are defined by duality. Whereas subsets of the poset in Figure 3 have greatest and least elements, the poset itself has neither a top nor a bottom element.

2.2. *Upper and Lower Bounds*

Let P be a poset and $S \quad P$. An element $x \quad P$ is an *upper bound* of S if $s \leq x$ for all $s \quad S$. A *lower bound* is defined by duality. The set of all upper bounds of S is denoted by S^* (read as "*S upper*") and the set of all lower bounds (or "*S lower*") is S_*. If S^* has a least element, it is called the *least upper bound* (l.u.b.) of S. By duality, if S_* has a largest element, it is called the *greatest lower bound* (g.l.b.) of S. A least upper bound or a greatest lower bound is always unique. For the least upper bound and the greatest lower bound of two elements x and y we write $x \quad y$ and $x \quad y$, respectively.

In some configurations a greatest lower bound or a least upper bound does not exist. This may be the case, if elements do not have common bounds or if a g.l.b. or l.u.b. does not exist.

2.3. *Lattices*

In the general case of a partially ordered set, we cannot expect that greatest lower bounds and least upper bounds always exist. Therefore, a more specific order structure is needed. A *lattice* L is a poset in which every pair of elements has a least upper bound and a greatest lower bound. A lattice is called *complete* when a greatest lower bound and a least upper bound exist for every subset of the poset. It can be proven that every finite lattice is complete. This is an important result, because it means that for every lattice with a finite number of elements we can always find least upper bounds and greatest lower bounds for every subset of the lattice.

Not every poset is a lattice, because posets exist in which not all subsets have greatest lower bounds and least upper bounds. It is, however, always possible to add elements to a poset to create a lattice. The process of the *normal completion* specifies how to find the smallest number of elements necessary to add to a poset to create a lattice, i.e., to build the minimal containing lattice of a poset (MacNeille 1937). The normal completion lattice has two important properties: (1) Every lattice is equal to its normal completion lattice. This means, whenever the poset is already a lattice, the normal completion does not add anything to the lattice. It leaves the lattice unchanged. (2) Since the normal completion turns any poset into a lattice, the successive normal completion provides the same normal completion lattice. Therefore, applying the normal completion more than once does not increase the number of elements added to the completion lattice. This implies that the number of elements in the completion lattice is bounded by 2^n for n elements in the poset.

2.4. *Direct Product*

There are many ways of generating new ordered sets from existing posets. One of these is the direct product of ordered sets. The *direct product* $P \times Q$ of two posets P and Q is the set of all pairs (x,y) with $x \quad P$ and $y \quad Q$, ordered by the rule that $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ in P and $y_1 \leq y_2$ in Q. The direct product of two lattices $L_1 \times L_2$ is a lattice with l.u.b. and g.l.b. formed as $(x_1, y_1) \quad (x_2, y_2) = (x_1 \quad x_2, y_1 \quad y_2)$ and $(x_1, y_1) \quad (x_2, y_2) = (x_1 \quad x_2, y_1 \quad y_2)$, respectively, for all $x_1, y_1 \quad L_1$, $x_2, y_2 \quad L_2$ and $(x_1, y_1), (x_2, y_2) \quad L_1 \times L_2$. The diagram of a direct product $P \times Q$ is drawn by replacing each point of the diagram of P by a copy of the diagram for Q, and connecting "corresponding" points.

3. Spatial Interpretation of the Characteristics of Posets and Lattices

The spatial relationship of areas being contained in other areas defines an order on the set of areas with the order relation ≤ corresponding to the notions of "is contained in", "lies in" or "is part of." For example, the crop zones and fields of Figure 1 represent a poset, because

- every area is contained within itself (reflexive),
- if an area x is contained in y and y is contained in x, then x and y are the same area (antisymmetric), and
- if an area x is contained in y which itself is contained in z, then x is also contained in z (transitive).

The dual to the order relation ≤ is read as "contains" or "consists of." This model corresponds to the concept of sets being a "subset of" or ("superset of") other sets. Therefore, we will also apply the set theoretic notions of union and intersection to spatial regions.

3.1. *Basic Relations*

Using the model of spatial regions ordered by "is contained in," the following questions about relationships between those regions can be answered by applying the characteristics of posets and lattices from Section 2 without any additional knowledge.

Question: Is region B contained in region A?
Answer: If $B \leq A$, then B is contained in A. If A and B are not comparable, then they either overlap or do not have any area in common.

Question: Which areas are contained in one or more given areas?
Answer: Determine all lower bounds of the given areas.

Question: Which areas contain one or more given areas?
Answer: Determine all upper bounds of the given areas.

Question: Which one is the largest area contained in one or more given areas?
Answer: Determine the greatest lower bound of the given areas.

Question: Which one is the smallest area containing one or more given areas?
Answer: Determine the least upper bound of the given areas?

3.2. *The Role of New Lattice Elements*

The use of greatest lower bounds and least upper bounds for answering some of these questions shows that we need a lattice in order to find an answer in all possible cases. This is why the normal completion is so important. It leaves us, however, with the question of how to interpret the newly created lattice elements. If the poset already had a top or bottom element, they are retained in the completion lattice. Otherwise a new top element (the whole area containing all other areas) and bottom element (the empty set or null area) are created.

More interesting are those new lattice elements other than the top or bottom element. The following property of the normal completion allows us to derive what they stand for:

> Let P be a poset and $A, B \subseteq P$. $(A^*)_*$ is then the set of the lower bounds of the upper bounds of A. This defines a *closure operator*, *C(A)*, on P such that $A \subseteq C(A)$; if $A \subseteq B$ then $C(A) \subseteq C(B)$; and $C(C(A)) = C(A)$. The family of $\mathrm{DM}(P) := \{A \subseteq P \mid (A^*)_* = A\}$ is a complete lattice when ordered by inclusion, in which greatest lower bound and least upper bound of a set of elements A_i are $\bigcap_{i \in I} A_i$ and $C(\bigcup_{i \in I} A_i)$, respectively.

From this follows that the new elements added to the poset to form a lattice can be interpreted as being the intersection of poset elements. For example, the lattice element X in Figure 4 is the spatial intersection of the crop zones A and B.

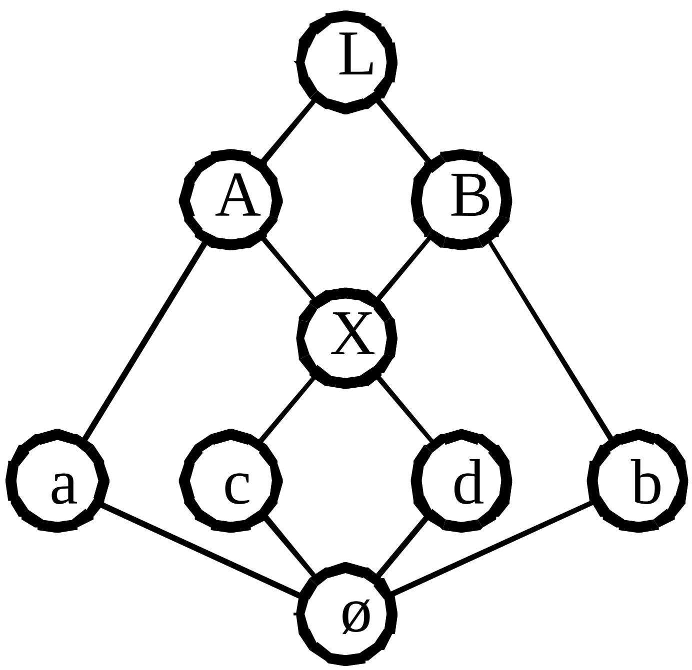

Figure 4: Lattice resulting from the normal completion of the poset in Figure 3.

3.2. *Direct Product and Overlay*

From Section 2.4 we know that the direct product of two lattices L_1 and L_2 is a lattice. The elements in the product lattice are pairs (a,b) where $a \in L_1$ and $b \in L_2$. Kainz (1988) suggested to replace (a,b) by $a \cap b$ and $a \cup b$ to model the intersection and union of the two orders represented by L_1 and L_2, respectively. This approach yields all possible intersections or unions between the elements of the two lattices in their correct position within the order.

If L_1 has n elements and L_2 has m elements then $L_1 \times L_2$ has $n \cdot m$ elements. The number of elements in the product lattice can be reduced depending on whether both lattices have the same top or bottom element. This is because the top element contains all other elements of the lattice and the bottom element is contained in all other lattice elements. Let t be the top element and b the bottom element. Then $x \cap t = x$, $x \cup t = t$ and $x \cap b = b$, $x \cup b = x$ for all x. Table 1 shows the number of elements in the product lattice for intersection and union. The result of these operations yields lattices that reflect all possible relationships between the spatial areas of two subdivisions after their union or intersection by using only set and order theory without the necessity for geometrical computations.

Table 1. Number of elements in the intersection and union of two lattices L_1 and L_2 with n and m elements depending on their top elements t_{L_1}, t_{L_2} and bottom elements b_{L_1}, b_{L_2}.

	Intersection	Union
$t_{L_1} = t_{L_2}$ and $b_{L_1} = b_{L_2}$	$(n-1)\ (m-1)+1$	$(n-1)\ (m-1)+1$
$t_{L_1} = t_{L_2}$ and $b_{L_1} \neq b_{L_2}$	$n\ \ m$	$(n-1)\ (m-1)+1$
$t_{L_1} \neq t_{L_2}$ and $b_{L_1} = b_{L_2}$	$(n-1)\ (m-1)+1$	$n\ \ m$
$t_{L_1} \neq t_{L_2}$ and $b_{L_1} \neq b_{L_2}$	$n\ \ m$	$n\ \ m$

4. Posets and Topological Spatial Relations

This section describes topological spatial operations based on the theory of Section 2. The operations are intersection, boundary, neighborhood, touch, and containment. Throughout this section, we use two intersecting squares as one simple example to illustrate each operation. The intersecting squares are represented as a simplicial complex. Simplicial complexes are a well-known concept in algebraic topology (Lefschetz 1975), that has also been used as a simple, but comprehensive framework to model spatial data in GIS (Frank and Kuhn 1986, Egenhofer *et al*. 1990, Pigot 1991, Worboys 1992). The methods as described here for simplices and simplicial complexes can be generalized to arbitrarily shaped cells and cell complexes built from such (Herring, 1987). Topology created by a simplicial subdivision of space is nearly equivalent to cellular topology, a popular way to represent topology in GISs, and it has the additional advantage of a rigorous and mathematically sound formalism (Herring, 1991). However, formalization and programming of simplices and simplicial complexes is easier as they are of fixed topological structure, whereas cells may have a varying number of faces in their boundaries. Readers who prefer the cellular model though, may think of the straight edges displayed in the geographic domain as being arbitrarily curved, but non-intersecting lines.

4.1. *Spatial Data Model*

A 2-dimensional simplicial complex is a collection of 0-, 1- and 2-dimensional simplices—points, segments, triangles—with certain characteristics. Every 1-simplex has two 0-dimensional faces and a 2-simplex has three 0-dimensional and three 1-dimensional faces. The intersection of two simplices is either empty or a face of both simplices. A simplicial complex defines a triangulation of the set of points of all simplices.

The two intersecting squares X and Y in Figure 5a may represent two land parcels. This configuration can be partitioned into a simplicial complex of points, segments and triangles (Figure 5b). Points are denoted with numbers, segments with lower case and triangles with upper case characters. Notice that the triangles form a partition over the intersecting squares. The elements of the partition (triangles) form building blocks out of which the higher level objects (areas) are built. This is an important fact which is useful in the spatial intersection operation.

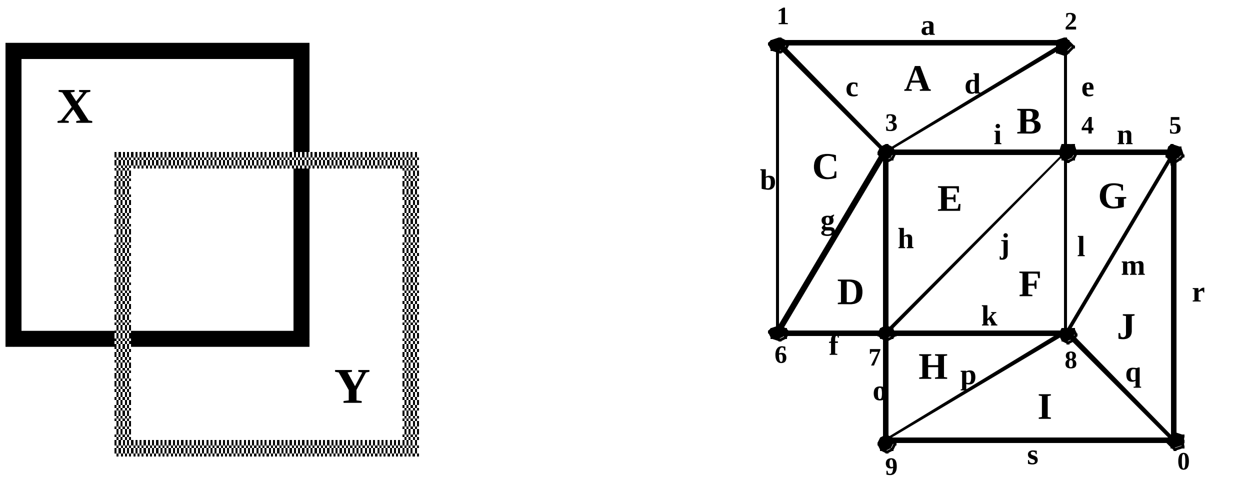

Figure 5: Two intersecting squares as model for topological operations (a) and their representation as a simplicial complex (b).

This model can be used to extend our definition of spatial inclusion by defining the relations between the elements of the simplicial complex such that:

- if the point p is an end point of the segment s, then $p \leq s$, and
- if the segment s is an edge of the triangle t, then $s \leq t$.

The simplices (points, segments, triangles) and areas define a poset (Figure 6a) which can be completed to a lattice (Figure 6b). This lattice will be used to demonstrate examples of topological operations on the lattice structure.

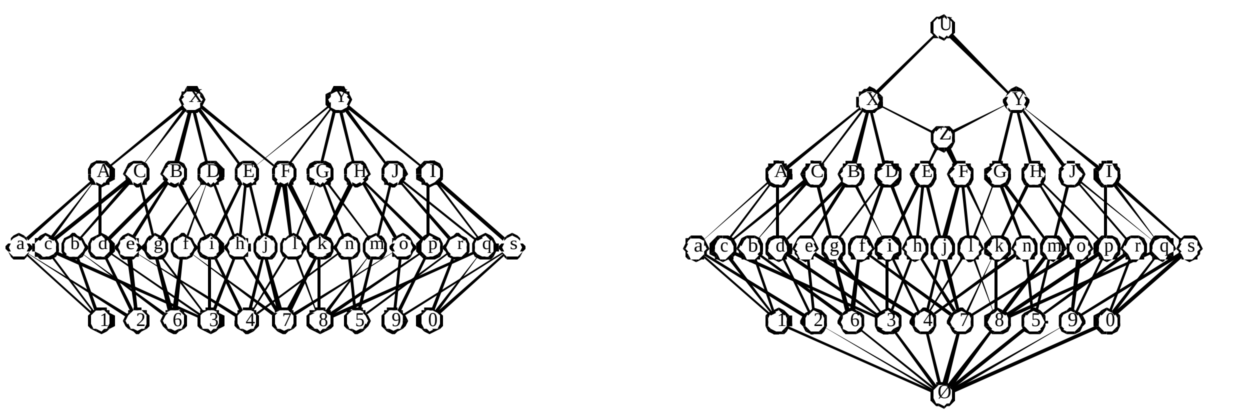

Figure 6: Poset of the simplices of two intersecting squares (a) and its normal completion (b).

The topological relationships between the simplices of the complex can now be expressed as order relations in the poset:

- Two points, p_1 and p_2, can share only one segment, which is $p_1 \quad p_2$.
- Two segments, s_1 and s_2, can share only one point, which is $s_1 \quad s_2$.
- Two segments, s_1 and s_2, can share only one triangle, which is $s_1 \quad s_2$.
- Two triangles, t_1 and t_2, can share only one segment, or one point, both of which is $t_1 \quad t_2$.

The fact that the point–segment–triangle part of the poset is already lattice-like might lead to the conclusion that the points and segments can be left out of the normal completion process. Greasley (1990) showed, however, that in general this is not possible and that all simplices must be included in the completion process.

4.2. *Intersection*

Proposition 4.1. The intersection of the elements of a set A is calculated as the greatest lower bound of A.

Proof. If the greatest lower bound was larger than the intersection, then at least one simplex of the partition would not be contained in every element of the set of areas A. But this is not the greatest lower bound of A. This means that the greatest lower bound cannot be larger than the intersection. If the greatest lower bound was smaller than the intersection, then at least one simplex of the partition that is in all the areas of the set A would not be in the greatest lower bound. But this cannot be the greatest lower bound. This means that the greatest lower bound cannot be smaller than the intersection. If the greatest lower bound of A can be neither smaller nor larger than the intersection of the elements of A then it must be equal to the intersection of the elements of A. □

Figure 7a shows the intersection of X and Y and figure 7b shows the corresponding operation on the Hasse diagram of the lattice, i.e., *intersection (X, Y)* = $X \quad Y$. The shaded element Z is the greatest lower bound and, therefore, the intersection of the elements X and Y.

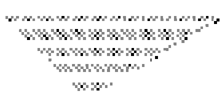

Figure 7: Intersection of two squares X and Y (a) and the corresponding operation in the lattice (b).

4.3. *Boundary*

The boundary of an area A can be determined by calculating the symmetric difference Δ of the elements that are covered by all the triangles of A, i.e., if C_2 is a simplicial 2-complex, then $boundary\,(C_2) = \Delta((C_2)_-)_-$.

The dark lines in Figure 8a show the boundary of the shaded area Z. Figure 8b displays the Hasse diagram and the boundary operation on the lattice.

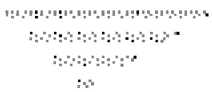

Figure 8: Boundary of area Z (a) and the corresponding operation in the lattice (b).

Area Z is a composition of the triangles E and F. The sets of elements that are covered by E and F are $\{i,h,j\}$ and $\{j,l,k\}$, respectively. The symmetric difference of these sets is the boundary:

$$boundary\,(Z) = \{i,h,j\}\,\Delta\{j,l,k\} = \{i,h,l,k\}.$$

4.4. *Neighborhood*

The neighborhood of an area A can be defined as all the triangles that share a segment with A: $\bigcup (a_-)^-$, a triangles(A). Let us call this neighborhood operation σ so that $\sigma(A)$ is the segment-neighborhood of A. We assume that A belongs to its neighborhood.

Figure 9a illustrates this definition. Triangles B and C each share a segment (thick lines) with the dark gray triangle A. Figure 9b shows the Hasse diagram and the operation on the lattice. The thick lines denote the search path. They lead from A to the set $\{a,c,d\}$, which is the cocover of A. They then lead from $\{a,c,d\}$ to the set $\sigma(A) = \{A,B,C\}$, which is the cover of $\{a,c,d\}$.

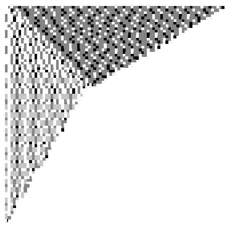

Figure 9: Segment neighbors of triangle A (a) and the corresponding operation in the lattice (b).

The neighborhood of an area A can also be defined as all the triangles that share a point with A: $\bigcup(((a_{-})_{-})^{-})^{-}$, a triangles(A). Let us call this neighborhood operation π so that $\pi(A)$ is the point-neighborhood of A. Again we assume that A belongs to its neighborhood. Figure 10a illustrates this definition. Each of the gray triangles B, C, D and E shares a point (highlighted) with the dark gray triangle A. Figure 10b shows the Hasse diagram and the corresponding operation on the lattice. The thick lines show the search path. They lead from A to the set $\{a,c,d\}$ to the set $\{1,2,3\}$. $\{1,2,3\}$ is the cocover of the cocover of A. They then lead from $\{1,2,3\}$ to the set $\{a,b,c,d,e,g,h,i\}$ to the set $\pi(A)=\{A,B,C,D,E\}$, which is the cover of the cover of $\{1,2,3\}$.

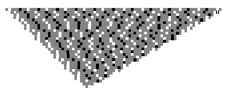

Figure 10: Point neighbors of triangle A (a) and the corresponding operation in the lattice (b).

Notice that $\sigma(\sigma(A)) \quad \pi(A)$. This can easily be shown. Consider a regular triangular tessellation of the plane. For every triangle A the segment-neighborhood $\sigma(A)$ can have at most 4 elements. $\sigma(\sigma(A))$ has at most 10 elements, whereas $\pi(A)$ has 13 elements.

A neighborhood operation can be performed several times for an area, written as $\pi^n(A)$, where n is the number of times the operation is to be applied. The factor n is the degree of neighborhood including neighbors that are farther away as n increases. This of course is not the ultimate definition of neighborhood. It does, however, take into account the information density, which influences our perception of neighborhood. It does not take into account the actual terrain. The town over the mountain or across the river may not be as close a neighbor as the town down the valley or down the same bank of the river.

4.5. *Touch*

The spatial relationship *touch* (Egenhofer and Franzosa 1991) can be derived from the set intersection, which was calculated with the greatest lower bound (see Section 4.2). If the greatest lower bound is a triangle or a set of triangles then the two areas intersect. If the greatest lower bound is a segment, a point, or a set of segments and/or points then the two areas touch. Notice that by this approach it is also possible to determine how the two areas touch. Figure 11a shows an example of this operation and Figure 11b shows how the operation is performed in the lattice. Element 8 is the greatest lower bound of the two elements Z and J, and represents the point where area Z and triangle J touch.

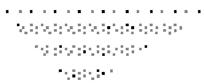

Figure 11: Two areas Z and J touch in a point (a) and the corresponding operation in the lattice (b).

4.6. *Containment*

As shown in Section 3.1 the upper bounds of a given set determine the containing areas, i.e., *containers* $(\{A, B\}) = \{A, B\}^*$. Figure 12a shows the triangles B and F and the areas they are contained in. In order to find these areas in the lattice we have to determine all upper bounds of the set of elements $\{B,F\}$ which gives $\{B,F\}^* = \{U, X\}$ (Figure 12b). It follows that the only areas that contain both triangles B and F are U and X.

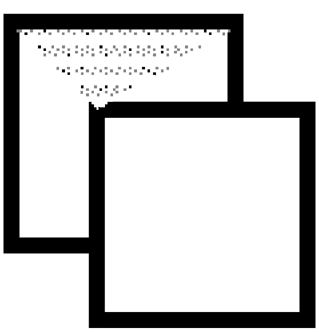

Figure 12: The only elements that contain both B and F are X and the whole area (a) and the operation of containment of B and F in U and X in the lattice (b).

5. Conclusion

In this paper we have presented a framework for relations between spatial objects based on the theory of ordered sets. It was demonstrated that relationships between spatial regions can be interpreted as order relations in posets and lattices. We have seen that the hierarchical model of a subdivision of land is not general enough to answer questions relating to the containment of spatial regions and should be replaced by the more general approach of ordered sets. The operations of cover, cocover, upper bounds, lower bounds, least upper bound, greatest lower bound, direct product, and normal completion of posets are not only useful in determining areas with specific characteristics and to model spatial overlay, but also provide tools to calculate topological operations of intersection, boundary, neighborhood, touch, and spatial containment.

Lattices can be used as a high-level model for spatial regions and the relations between them, built on the topological representation of these areas as a simplicial complex. Some queries can then already be answered by simple lattice operations without having to perform topological operations or geometric calculations. In addition the lattice representation is also a model for topological operations.

Having both ways to represent spatial relations—the topological and the order theoretic approach—allows us to look for the most efficient model to answer certain spatial queries. If, for instance, we want to know whether a given region is contained in another one, we can choose the topological model and look at the intersections of the interiors and boundaries of both regions. If the interiors intersect and the boundaries do not, then we know that one area is inside the other. We may also look into a poset representation of the regions and check whether both areas are comparable. If they are, then we know that one area is contained in the other. What we have done is to map a query either to the category of topological spaces or to the category of ordered sets and find the answer in the respective model. This can be done, because there is a correspondence between the topological and the poset model of spatial regions.

One of the crucial parts of the poset model is the normal completion. The classical Dedekind-MacNeille procedure uses the powerset for the completion of a poset to a lattice. Since real world posets of spatial regions usually consist of hundreds or even thousands of elements the classical algorithm cannot be applied. Although alternate approaches have been proposed (Perry 1990) a thorough analysis and implementation of these algorithms is still missing. It might also be worthwhile to look into the characteristics of posets and lattices representing spatial subdivisions to find out whether certain types of lattices can be identified. This could influence the normal completion process and allow for the design of tailored, efficient normal completion algorithms.

Acknowledgments

Andrew Frank, Alan Saalfeld, and Bob Franzosa stimulated the research with valuable comments and discussions. Their help and support are gratefully acknowledged. This work was partially supported by grants from the U.S. Bureau of the Census under joint statistical agreements 87–11 and 89–33. Additional support came from Intergraph Corporation and from NSF for the NCGIA under grant number SES-88-10917.

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