

# A Model for Detailed Binary Topological Relationships\*

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## Abstract

A model of binary topological relationships between  $n$ -dimensional spatial objects is extended to account for detailed spatial relationships. Binary topological relationships are defined in terms of topological invariants of the four intersections between the set-theoretic boundary and interior of two homogeneously  $n$ -dimensional spatial objects. In the original model, these intersections were analyzed according to their content, i.e., emptiness/non-emptiness, identifying eight topological relationships that can be realized between two  $n$ -dimensional objects embedded in  $\mathbb{R}^n$ . It is shown that more details about these topological relationships can be expressed by considering the dimension and the number of disconnected segments for non-empty intersections. With each of these topological invariants, we find refinements for four of the eight initial topological relations from empty/non-empty intersections.

## 1 Introduction

Languages to interact with information systems about geographic space for planning, navigation, and way finding make extensive use of spatial predicates (Davis 1986, Kuipers and Levitt 1988, Egenhofer 1990). Spatial predicates are ingredients of any spatial query language (Frank 1982, Herring *et al.* 1988, Egenhofer 1991) and their formal definitions are necessary to process such queries in a spatial database (Egenhofer 1989b). The conceptual complexity of these terms and the lack of appropriate formalizations have been widely acknowledged (Abler 1987, NCGIA 1989, Guenther and Buchmann 1990) and studies have been undertaken in a variety of areas. Linguists and cognitive scientists have studied the use of spatial predicates in natural language (Talmy 1983, Herskovits 1986, Retz-Schmidt 1988), while computer scientists, mathematicians, engineers, and geographers have concentrated their efforts on the development of formalisms to represent the geometric aspects of spatial predicates, called *spatial relationships* (Peuquet and Ci-Xiang 1987, Chang *et al.* 1989, Robinson 1990, Hernández 1991, Freksa 1992, Jungert 1992).

The overall goal is to find a way of distinguishing among several topological relationships by other than purely intuitive means. In the past, formalisms for spatial relationships have emphasized

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directions such as the geographic concepts of north, south, east, and west—and their refinements into north-east, north-west, etc. (Peuquet and Ci-Xiang 1987, Dutta 1988, Hernández 1991, Frank 1992)—or the corresponding image-oriented direction relations above/below/right/left (Freeman 1975, Chang *et al.* 1987). At the same time, only few efforts have been put into the investigations of spatial relationships that are independent of the orientation of the objects with respect to a reference frame or reference objects.

This paper focuses on the formalization of *topological relationships*, i.e., spatial relationships that are invariant under bijective and continuous transformations that have also continuous inverses. Topological equivalence does not necessarily preserve distances and directions. Instead, topological notions include continuity, interior, and boundary, which are defined in terms of neighborhood relations. If topological aspects have been part of previous investigations, the definitions of topological relationships have been based upon, or mixed with, other concepts such as metric (Peuquet 1986) or order (Jungert 1988, Chang *et al.* 1989, Lee and Hsu 1990). Spatial relationships derived from distances, directions, and their logical combinations (Peuquet 1986) are based upon the concept of *metric* so that topology is derived from metric. Conceptually, this distinction is insufficient to distinguish between inside and outside, and any implementation thereof underlies the limitations of finite computers and their arithmetic operations. Another formalization of topological relationships uses symbolic projections of the objects onto perpendicular axes and encodes the objects as strings of segments (Chang *et al.* 1989, Jungert 1988, Lee and Hsu 1990). While this method uses symbolic processing, it is not invariant under rotation of the objects, i.e., different 2D strings may be produced for the same topological relationship if the two objects are rotated against the projection axis. Therefore, a much larger amount of relations is necessary to describe a complete set (Lee and Hsu 1992).

In this paper, a novel approach is pursued to model binary topological relationships with purely topological means. Binary topological relationships are defined in terms of topological invariants of the four intersections between the set-theoretic boundary and interior of two homogeneously  $n$ -dimensional spatial objects. In the original model, these intersections were analyzed according to their content, i.e., emptiness/non-emptiness, identifying eight topological relationships that can be realized between two  $n$ -dimensional objects embedded in  $\mathbb{R}^n$  (Egenhofer and Franzosa 1991). The main use of this theory is in query language design, as it may serve as a backbone for the definition of terms used in spatial query languages (Mark and Egenhofer 1992) and for processing spatial queries (Egenhofer 1993). Compared to previously proposed approaches to describe topological relationships, our formalism is the most powerful and complete, and has gained much popularity in the area of GIS and spatial databases (Herring 1991, Pigot 1991, Svensson and Zhaxue 1991, Hadzilacos and Tryfona 1992, Hazelton *et al.* 1992, de Hoop and van Oosterom 1992, Smith and Park 1992). While this model is sufficient to distinguish among the major categories of topological relations, it cannot identify differences that are sometimes relevant. For example, in some U.S. states, the definition of a neighbor parcel requires the two parcels to share at least one edge. In other states, a neighbor needs only a common corner point. In order to make such distinctions of detailed topological relations, we will extend the model of empty/non-empty intersections to consider other topological invariants such as the dimension of an intersection and the number of separations in an intersection.

A number of generalizations can be made: (1) The concepts, developed for 2-dimensional complexes that are imbedded in  $\mathbb{R}^2$ , generalize to other spatial data models such as 2-cells or non-convex sets. (2) The methods and tools (boundary and interior operations, dimension, separation) are formalized for simplicial complexes, but can be found in other spatial data models as well. They are shape-independent. (3) Though the examples in this paper will refer to 2-dimensional simplicial complexes in  $\mathbb{R}^2$ , the proposed method generalizes to  $n$ -complexes in  $\mathbb{R}^n$ , with  $n \geq 2$ . The 0-

dimensional case is trivial and the particular characteristics of the 1-dimensional case, which deviate from this model, have been treated elsewhere (Pullar and Egenhofer 1988, Egenhofer and Franzosa 1991).

The remainder of this paper is organized as follows: Section 2 motivates the concept of topological relationships, introduces a model of binary topological relationships based upon the four intersections of boundaries and interiors, and demonstrates its shortcomings to express detailed topological relations. Section 3 introduces a topological data model for finite sets, which is used to derive and define formally the binary topological relationships between two  $n$ -complexes from empty and non-empty boundary and interior intersections. These topological relationships are refined in Section 4 where the dimension and the Euler-Poincaré characteristic are used as topological invariants to evaluate the intersections. For all classes of topological relationships geometric interpretations are given in the form of prototypical examples between two 2-dimensional complexes. Section 5 presents the conclusions.

## 2 Topological Relationships

Figure 1 shows a motivating example upon which the phenomena of topological relationships will be explained. Initially, the two objects,  $A$  and  $B$ , are such that humans would use terms like *overlap* or *intersect* to describe their relationship (Figure 1a). A particular characteristic of this topological relationship is the relation among their boundaries and interiors, i.e., the boundaries coincide in two points, the boundary of each object runs through the other's interior, and both interiors coincide partially. The topological relationship between the two objects stays the same as long as these characteristics are preserved (Figure 1b); however, it changes once the common parts are the only coinciding boundary parts, while the interiors have nothing in common with the other object's parts (Figure 1c). In a similar configuration (Figure 1d) the two objects have one boundary edge less in common; however, this difference does not influence the judgment about the topological relationship between the two objects and humans would continue to use the term *touch* or *meet* to describe the relationship. Figure 1e shows the two objects after a further transformation that significantly changed the topological relationship so that the two objects do not *touch* anymore, but they are *disjoint* from each other. Even if  $B$  is moved further away from  $A$ , this topological relationship is the same (Figure 1f).

These observations led to a comprehensive model of binary topological relationships (Egenhofer 1989a, Egenhofer and Herring 1990, Egenhofer and Franzosa 1991), representing the topological relationship  $R_x$  between two spatial objects,  $A$  and  $B$ , as the intersections  $I$  of the boundary and interior of  $A$  with the boundary and interior of  $B$ . Boundary and interior can be combined to form the four fundamental criteria of the topological relationships between two spatial objects, called the *4-intersection*. These are: (1) common boundaries, denoted by  $\partial \cap \partial$ , (2) common interiors ( ${}^\circ \cap {}^\circ$ ), (3) boundary parts of one object that coincide with interior parts of the other ( $\partial \cap {}^\circ$ ), and (4) interior parts of one object that coincide with boundary parts of the other ( ${}^\circ \cap \partial$ ). Figure 2 displays a 4-intersection and depicts a graphical interpretation of its topological relationship, emphasizing the four intersections.

In order to study the binary topological relationships  $R_x$  between  $A$  and  $B$  it is sufficient to examine the topological invariants of the intersections of the boundaries and interiors of  $A$  and  $B$ . For this purpose, the content of the intersections is selected as the fundamental criterion for the distinction of topological relationships, because it describes a closed set of relationships with complete coverage. With the binary values *empty* ( $\emptyset$ ) and *non-empty* ( $-\emptyset$ ),  $2^4$  different specifications are given, which provide the basis for the formal definition of the topological relationships. In (Egen-

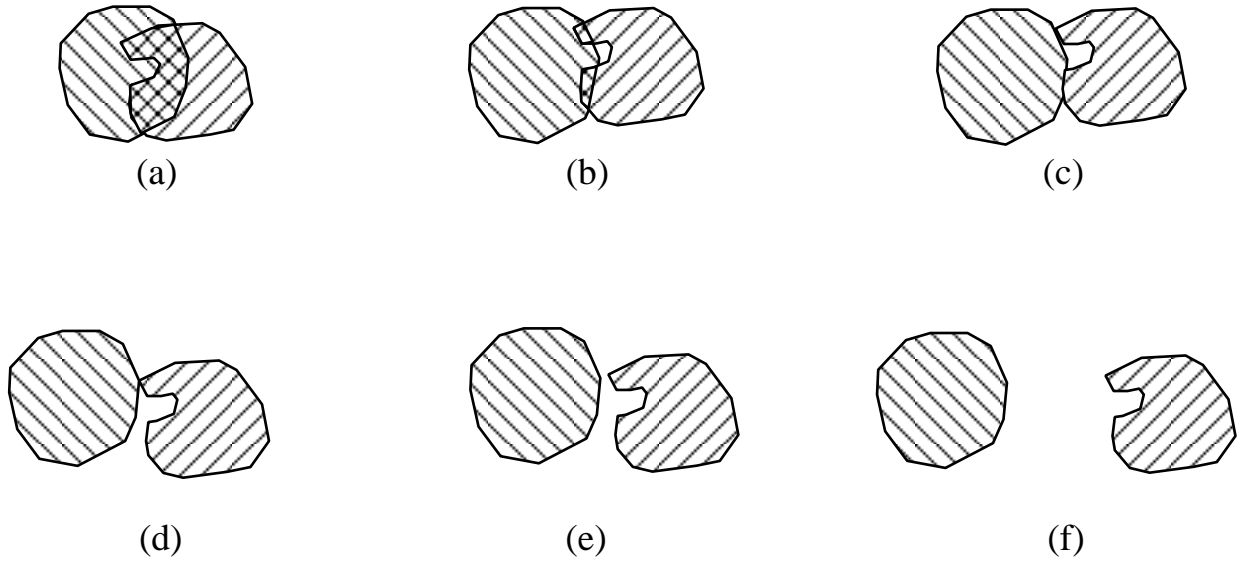


Figure 1: Examples of binary topological relationships.

hofer and Franzosa 1991) we showed that only eight of these relations can be realized between two regions if (1) both objects have connected boundaries, (2) are homogeneously 2-dimensional, and (3) embedded in  $\mathbb{R}^2$ . They are:

**Definition 2.1** *If all four intersections among all faces are empty then the two regions are disjoint (Figure 3a), i.e.,*

$$R_{\text{disjoint}}(A, B) \Leftrightarrow I_{\emptyset}(A, B) = \begin{pmatrix} \emptyset & \emptyset \\ \emptyset & \emptyset \end{pmatrix} \quad (1)$$

**Definition 2.2** *If the intersection between the boundaries is not empty, whereas all other 3 intersections are empty, then the two regions meet (Figure 3b), i.e.,*

$$R_{\text{meet}}(A, B) \Leftrightarrow I_{\emptyset}(A, B) = \begin{pmatrix} \neg\emptyset & \emptyset \\ \emptyset & \emptyset \end{pmatrix} \quad (2)$$

**Definition 2.3** *Two regions are equal if both intersections of boundary and interior are not empty, while the two boundary-interior intersections are empty (Figure 3c), i.e.,*

$$R_{\text{equal}}(A, B) \Leftrightarrow I_{\emptyset}(A, B) = \begin{pmatrix} \neg\emptyset & \emptyset \\ \emptyset & \neg\emptyset \end{pmatrix} \quad (3)$$

**Definition 2.4** *A region A is inside of another region B if (1) A and B share the same interior, but not the boundary, (2) if A's boundary is a subset of B's interior, and (3) nothing of B's boundary intersects with any part of A's interior (Figure 3d), i.e.,*

$$R_{\text{inside}}(A, B) \Leftrightarrow I_{\emptyset}(A, B) = \begin{pmatrix} \emptyset & \neg\emptyset \\ \emptyset & \neg\emptyset \end{pmatrix} \quad (4)$$

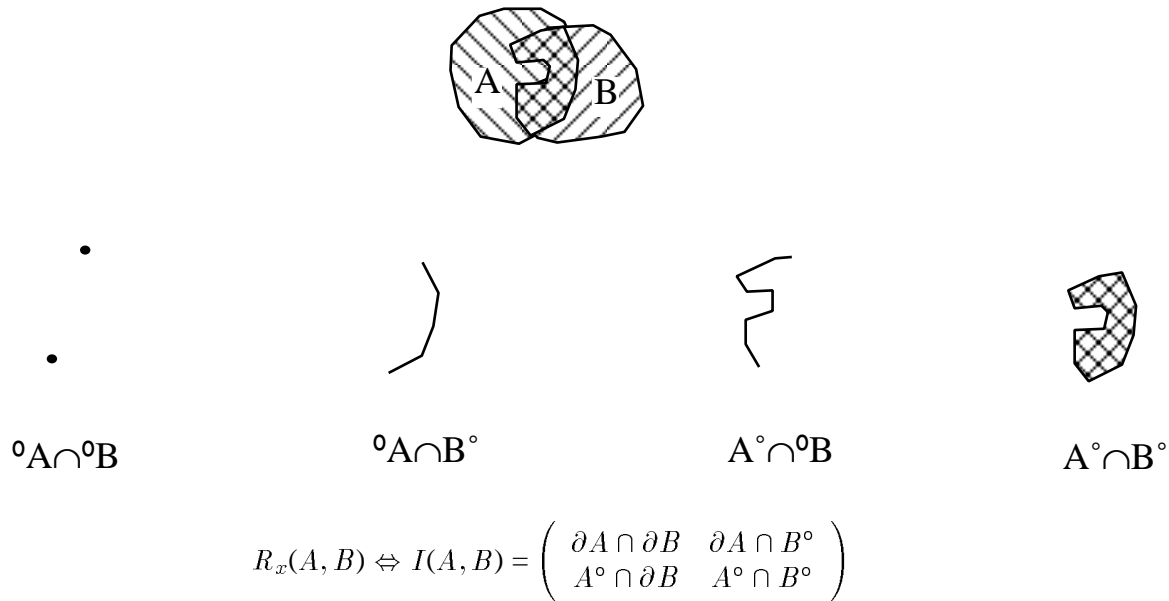


Figure 2: Comparing the boundaries and interiors of two overlapping objects.

**Definition 2.5** A region  $A$  contains another region  $B$  if  $A$  and  $B$  share the same interior, but nothing of their boundaries is in common; if  $B$ 's boundary is a subset of  $A$ 's interior, and nothing of  $A$ 's boundary intersects with any piece of  $B$ 's interior (Figure 3e), i.e.,

$$R_{\text{contains}}(A, B) \Leftrightarrow I_{\emptyset}(A, B) = \begin{pmatrix} \emptyset & \neg\emptyset \\ \emptyset & \neg\emptyset \end{pmatrix} \quad (5)$$

**Definition 2.6** A region  $A$  is coveredBy another region  $B$  if both regions have common parts in their boundaries and interiors; if parts of  $A$ 's interior intersect with parts of  $B$ 's boundary; and if nothing of  $B$ 's interior is part of  $A$ 's boundary (Figure 3f), i.e.,

$$R_{\text{coveredBy}}(A, B) \Leftrightarrow I_{\emptyset}(A, B) = \begin{pmatrix} \neg\emptyset & \neg\emptyset \\ \emptyset & \neg\emptyset \end{pmatrix} \quad (6)$$

**Definition 2.7** A region  $A$  covers another region  $B$  if both regions share some common boundary and interior; if  $B$ 's interior intersects with  $A$ 's boundary; and if none of  $A$ 's interior is part of  $B$ 's boundary (Figure 3g), i.e.,

$$R_{\text{covers}}(A, B) \Leftrightarrow I_{\emptyset}(A, B) = \begin{pmatrix} \neg\emptyset & \emptyset \\ \neg\emptyset & \neg\emptyset \end{pmatrix} \quad (7)$$

**Definition 2.8** Two regions overlap if they have common boundaries and interiors, and both boundaries intersect with the opposite interiors (Figure 3h), i.e.,

$$R_{\text{overlap}}(A, B) \Leftrightarrow I_{\emptyset}(A, B) = \begin{pmatrix} \neg\emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset \end{pmatrix} \quad (8)$$

In order to define these operations formally, we use the *simplex data model*.

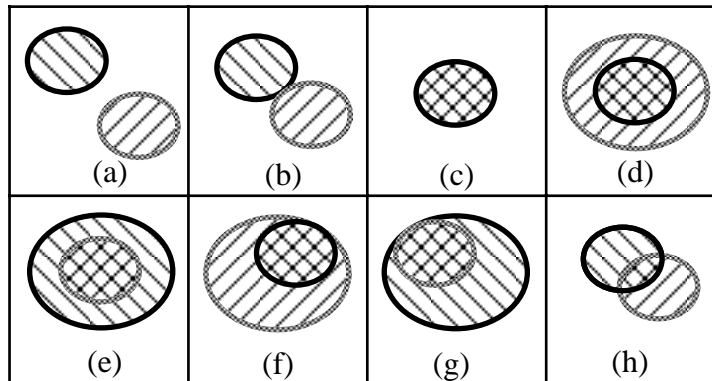


Figure 3: Examples of the eight basic topological relationships between two spatial regions with connected boundaries embedded in  $\mathbb{R}^2$ .

### 3 A Topological Data Model

This section presents a data model upon which the concepts of boundary, interior, and their intersections will be formally described (Spanier 1966, Munkres 1966). It is based upon the mathematical branch of combinatorial topology, in which a sophisticated method has been developed to classify and describe formally point sets. This theory has been used for modeling spatial data (Corbett 1979, White 1984) and was applied to spatial data models in geographic information systems (Frank and Kuhn 1986, Herring 1987, Worboys 1992), both for two-dimensional (Egenhofer 1987) and three-dimensional (Carlson 1987) geometry. The entire  $n$ -dimensional space, the universe  $U$ , is represented by a single  $n$ -dimensional simplicial complex so that (1) every complex  $c_n$  is connected and (2) every face of  $c_n$  of dimension 0 through  $n-1$  is a face of an  $n$ -simplex of  $c_n$  (Frank and Kuhn 1986). Objects in this space are represented as sub-complexes of  $U$ . In geographical terms, such a decomposition results in an organization of space based upon irregularly shaped polygons for surface modeling or polyhedra for solids modeling (Egenhofer and Herring 1991).

#### 3.1 Simplex and Simplicial Complex

Spatial objects are classified according to their spatial dimensions. For each dimension, a minimal object exists, called *simplex*. An  $n$ -simplex is the convex hull of  $n+1$  geometrically independent points, called *vertices*. Examples of minimal spatial objects are 0-simplices representing nodes, 1-simplices, which stand for edges, 2-simplices for triangles, 3-simplices for tetrahedrons, etc. Any  $n$ -simplex is bounded by  $n+1$  geometrically independent simplices of dimension  $n-1$ . For example,

a triangle, a 2-simplex, is bounded by three 1-simplices. These 1-simplices are geometrically independent if no two edges are parallel and no edge is of length 0. A face of a simplex  $S$  is any simplex that is contained in  $S$ . The term  $n$ -face denotes a face of dimension  $n$ .

An *orientation* of a simplex fixes the vertices to lie in a sequence. The orientation of a 0-simplex is unique; the two orientations of a 1-simplex can be interpreted as the direction *from* vertex  $A$  to vertex  $B$  and reverse *from*  $B$  to  $A$ ; the orientations of a 2-simplex can be interpreted as *clockwise* or *counterclockwise*. An ordered  $n$ -simplex  $s_n$  may then be represented by its vertices in the following form:

$$s_n = \langle x_0, \dots, x_n \rangle \tag{9}$$

A *simplicial complex*, or briefly *complex*, is a finite collection of simplices and their faces such that if the intersection between two simplices of this collection is not empty, then the intersection is a simplex that is a face of both simplices.

### 3.2 Algebraic Boundary

The *algebraic boundary*, denoted by  $\underline{\partial}$ , consists of all  $n-1$ -faces that bound an  $n$ -simplex. For an ordered  $n$ -simplex,  $s_n$ , its algebraic boundary is determined by

$$\underline{\partial}s_n = \sum_{i=0}^n (-1)^i \langle x_0, \dots, \widehat{x}_i, \dots, x_n \rangle \tag{10}$$

where  $\widehat{x}_i$  denotes that the face  $x_i$  is to be omitted. The bounding  $n-1$ -simplices form a chain, which is an element of a free Abelian (i.e., additive) group, with  $-\langle x_0, \dots, x_n \rangle = \langle x_n, \dots, x_0 \rangle$  and  $\langle x_0, \dots, x_n \rangle - \langle x_0, \dots, x_n \rangle = 0$ . Hence, the algebraic boundary of a simplicial complex  $c_n$  can be determined as the sum of the algebraic boundaries of all its simplices  $s_n$ .

$$\underline{\partial}c_n = \sum \underline{\partial}s_n \in c_n \tag{11}$$

The algebraic boundary operation upon an  $n$ -complex considers only faces of dimension  $n-1$ . While this operation is sufficient to treat the topological relationships among 1-complexes, it impedes a more general treatment of topological relationships, which are sometimes based upon common faces of dimension  $n-2$  or less and thus cannot be used immediately for the general specification of topological relationships. Figure 4 shows an example of a topological relationship for which the use of the algebraic boundary fails. The intersection of the boundaries of the 2-complex in one 0-dimensional face is a crucial property of this topological relationship; however, the intersection of the two algebraic boundaries does not identify any common faces and applying the boundary operation upon the intersection does not help, because boundary applied twice is always zero.

### 3.3 Set-Theoretic Boundary

To overcome these shortcomings, the *set-theoretic boundary* is introduced. It considers all faces in the boundary, from dimension  $n-1$  to 0. Its definition is based upon the algebraic boundary (Equation 10) and the skeleton (Equation 12). The  $r$ -skeleton of a complex  $c_n$ , denoted by  $c_n^{(r)}$ , is defined as the union of all its simplices of dimension at most  $r$ .

$$c_n^{(r)} = \bigcup_{i=0}^r s_i \in c_n \tag{12}$$

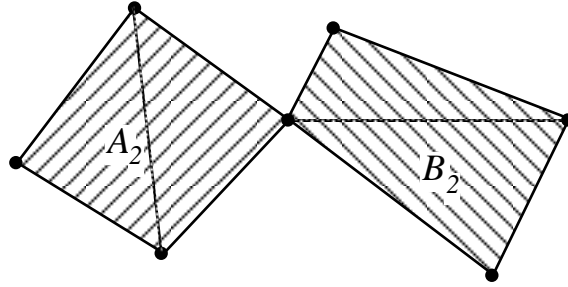


Figure 4: Two neighboring 2-complexes,  $A_2$  and  $B_2$ , sharing only a common node.

The set-theoretic boundary of an  $n$ -dimensional complex  $c_n$ , denoted  $\partial c_n$ , is defined as the  $n$ -1-skeleton of the algebraic boundary of  $c_n$ .

$$\partial c_n = \bigcup_{r=0}^{n-1} c_n^{(r)} \in \underline{\partial} c_n \quad (13)$$

An alternative definition of the set-theoretic boundary (with the same result) applies the algebraic boundary operator to the  $n$ -complex, and then takes the union of the algebraic boundary applied again to each element of the  $(n-1)$ -bounding cycle(s) (Corbett 1985).

The *interior* of an  $n$ -dimensional complex  $c_n$ , denoted  $c_n^\circ$ , is the set of all faces of the  $n$ -skeleton of  $c_n$  that are not part of the set-theoretic boundary.

$$c_n^\circ = c_n^{(n)} - \partial c_n \quad (14)$$

Subsequently, the term *boundary* will always refer to the set-theoretic boundary.

### 3.4 Topological Relations between Simplicial Complexes

Boundary and interior are sets upon which the traditional operations of set theory apply. In this context, only the set intersection will be needed so that the topological relationships between two simplicial complexes can be implemented as the set intersections of their boundaries and interiors.

The following example demonstrates how the previously introduced concepts are applied to determine the topological relationship between two 2-complexes. Let  $A_2$  and  $B_2$  (Figure 5) be the two 2-complexes in Equations (15) and (16), respectively.

$$A_2 = \{\langle N4, N3, N2 \rangle, \langle N4, N3 \rangle, \langle N3, N2 \rangle, \langle N2, N4 \rangle, \langle N2 \rangle, \langle N4 \rangle, \langle N3 \rangle\} \quad (15)$$

$$B_2 = \{\langle N4, N3, N2 \rangle, \langle N1, N3, N4 \rangle, \langle N1, N4, N2 \rangle, \langle N1, N3 \rangle, \langle N1, N4 \rangle, \langle N1, N2 \rangle, \langle N4, N3 \rangle, \langle N4, N2 \rangle, \langle N3, N2 \rangle, \langle N1 \rangle, \langle N2 \rangle, \langle N3 \rangle, \langle N4 \rangle\} \quad (16)$$

Their boundaries and interiors (Equations 17–20) are calculated according to Equations (13) and (14), respectively.

$$\partial A_2 = \{\langle N2, N4 \rangle, \langle N3, N4 \rangle, \langle N2, N3 \rangle, \langle N2 \rangle, \langle N3 \rangle, \langle N4 \rangle\} \quad (17)$$

$$A_2^\circ = \{\langle N4, N3, N2 \rangle\} \quad (18)$$

$$\partial B_2 = \{\langle N2, N3 \rangle, \langle N1, N3 \rangle, \langle N1, N2 \rangle, \langle N1 \rangle, \langle N2 \rangle, \langle N3 \rangle\} \quad (19)$$

$$B_2^\circ = \{\langle N4, N3, N2 \rangle, \langle N1, N3, N4 \rangle, \langle N1, N4, N2 \rangle, \langle N1, N4 \rangle, \langle N3, N4 \rangle, \langle N2, N4 \rangle, \langle N4 \rangle\} \quad (20)$$

The calculations of the four intersections of their boundaries and interiors (Equations 21–24) are straightforward set intersections.

$$\partial A_2 \cap \partial B_2 = \{\langle N2, N3 \rangle, \langle N2 \rangle, \langle N3 \rangle\} \quad (21)$$

$$\partial A_2 \cap B_2^\circ = \{\langle N3, N4 \rangle, \langle N4, N2 \rangle, \langle N4 \rangle\} \quad (22)$$

$$A_2^\circ \cap \partial B_2 = \{\} \quad (23)$$

$$A_2^\circ \cap B_2^\circ = \{\langle N2, N3, N4 \rangle\} \quad (24)$$

Finally, the analysis for emptiness/non-emptiness is trivial (Equation 25).

$$I_\emptyset(A_2, B_2) = \begin{pmatrix} \neg\emptyset & \neg\emptyset \\ \emptyset & \neg\emptyset \end{pmatrix} \quad (25)$$

It reveals that the specification of the topological relationship between  $A_2$  and  $B_2$  is identical with specification (6), so that  $A_2$  is *coveredBy*  $B_2$ .

Table 1: The eight specifications of topological relationships between two connected  $n$ -complexes.

	$\partial A_n \cap \partial B_n$	$\partial A_n \cap B_n^\circ$	$A_n^\circ \cap B_n^\circ$	$A_n^\circ \cap \partial B_n$
disjoint	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
meet	$\neg\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
equal	$\neg\emptyset$	$\emptyset$	$\neg\emptyset$	$\emptyset$
inside	$\emptyset$	$\neg\emptyset$	$\neg\emptyset$	$\emptyset$
contains	$\emptyset$	$\emptyset$	$\neg\emptyset$	$\neg\emptyset$
coveredBy	$\neg\emptyset$	$\neg\emptyset$	$\neg\emptyset$	$\emptyset$
covers	$\neg\emptyset$	$\emptyset$	$\neg\emptyset$	$\neg\emptyset$
overlap	$\neg\emptyset$	$\neg\emptyset$	$\neg\emptyset$	$\neg\emptyset$

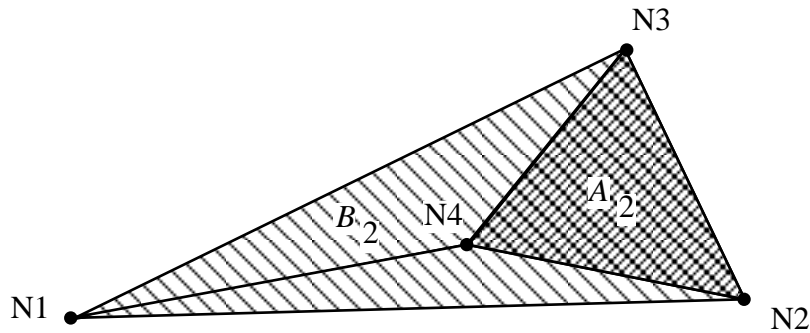


Figure 5: Two 2-complexes such that  $A_2$  is *coveredBy*  $B_2$ .

## 4 Detailed Topological Relationships

### 4.1 Details about Dimensions

The topological invariant of the *dimension* can be applied to any of the four intersections such that  $I_{\emptyset, dim}$  describes the corresponding relationships in terms of the content ( $\emptyset$ ) and the dimension (*dim*) of the four intersections.

The dimension of the empty set is -1, i.e., constant, while the dimensions of non-empty intersections are based upon the dimension of the simplices that are elements in the intersections. The dimension of a simplex, denoted by  $dim(s)$ , is one less than the number of its vertices (Equation 26).

$$dim(\langle \underbrace{A, \dots, N}_n \rangle) = n - 1 \quad (26)$$

The dimension of a non-empty complex  $c_n$ , denoted by  $dim(c_n)$ , is taken to be the maximum of the dimensions of the simplices of  $c_n$  (Equation 27).

$$dim(c_n) = sup(dim(s_i \in c_n)) \quad (27)$$

Likewise, the dimension of the boundary  $\partial$  of a complex  $c_n$ , denoted by  $dim(\partial c_n)$ , is defined to be the maximum of the dimensions of all faces in  $\partial c_n$ , i.e.,  $dim(\partial c_n)$  is -1 if  $\partial c_n$  is empty; otherwise it is  $n-1$ . In analogy to the dimension of the boundary, the dimension of the interior of a non-empty complex,  $dim(c_n^\circ)$ , is  $n$ .

The dimensions of the four non-empty intersections of boundaries and interiors are in the range between 0 and the maximum of the dimensions of the corresponding intersecting faces, i.e., between 0 and  $n-1$  for intersections with boundaries, and between 0 and  $n$  for  $A^\circ \cap B^\circ$ . For example, the dimensions of the four intersections (21–23) for the two 2-complexes in Figure 5 are:

$$dim(\partial A_2 \cap \partial B_2) = sup(dim(\langle N2, N3 \rangle), dim(\langle N2 \rangle), dim(\langle N3 \rangle)) = 1 \quad (28)$$

$$dim(\partial A_2 \cap B_2^\circ) = sup(dim(\langle N3, N4 \rangle), dim(\langle N4, N2 \rangle), dim(\langle N4 \rangle)) = 1 \quad (29)$$

$$dim(A_2^\circ \cap \partial B_2) = sup(dim(\langle \rangle)) = -1 \quad (30)$$

$$dim(A_2^\circ \cap B_2^\circ) = sup(dim(\langle N2, N3, N4 \rangle)) = 2 \quad (31)$$

If the two  $n$ -complexes have codimension 0, the ranges of the dimensions of the non-empty intersections are limited such that only  $\partial \cap \partial$  distinguishes different dimensions (Equation 32), while the dimensions of the other intersections depend exclusively upon the dimensions of the two complexes (Equations 33–35); therefore, all relationships with non-empty  $\partial \cap \partial$  are candidates for refinements due to different dimensions.

$$dim(\partial A_n \cap \partial B_n) = 0 \dots (n - 1) \quad (32)$$

$$dim(\partial A_n \cap B_n^\circ) = n - 1 \quad (33)$$

$$dim(A_n^\circ \cap \partial B_n) = n - 1 \quad (34)$$

$$dim(A_n^\circ \cap B_n^\circ) = n \quad (35)$$

A particular role takes  $R_{\text{equal}}$  whose  $\partial \cap \partial$  is non-empty, but nevertheless unique, because it is defined with respect to the dimension of the two  $n$ -complexes, i.e., always  $n-1$  (Equation 36); therefore, only one specification exists for  $R_{\text{equal}}$  and it is impossible to describe different types of  $R_{\text{equal}}$  with different dimensions.

$$I_{\emptyset, dim}(A_n, B_n) = \begin{pmatrix} \neg \emptyset & \emptyset \\ dim=n-1 & \emptyset \\ \emptyset & \neg \emptyset \end{pmatrix} \equiv \begin{pmatrix} \neg \emptyset & \emptyset \\ \emptyset & \neg \emptyset \end{pmatrix} \quad (36)$$

The non-empty  $\partial \cap \partial$  of the other specifications are variable in the range between 0 and  $n-1$  (Equations 37–40).

$$R_{dim\text{-meet}}(A_n, B_n) \Leftrightarrow I_{\emptyset, dim}(A_n, B_n) = \begin{pmatrix} \neg\emptyset & \emptyset \\ dim=0\dots(n-1) & \emptyset \end{pmatrix} \quad (37)$$

$$R_{dim\text{-covers}}(A_n, B_n) \Leftrightarrow I_{\emptyset, dim}(A_n, B_n) = \begin{pmatrix} \neg\emptyset & \emptyset \\ dim=0\dots(n-1) & \neg\emptyset \end{pmatrix} \quad (38)$$

$$R_{dim\text{-coveredBy}}(A_n, B_n) \Leftrightarrow I_{\emptyset, dim}(A_n, B_n) = \begin{pmatrix} \neg\emptyset & \neg\emptyset \\ dim=0\dots(n-1) & \neg\emptyset \end{pmatrix} \quad (39)$$

$$R_{dim\text{-overlap}}(A_n, B_n) \Leftrightarrow I_{\emptyset, dim}(A_n, B_n) = \begin{pmatrix} \neg\emptyset & \neg\emptyset \\ dim=0\dots(n-1) & \neg\emptyset \end{pmatrix} \quad (40)$$

Figure 6 shows examples for the different relationships between 2-complexes in  $\mathbb{R}^2$ , which can be distinguished by the empty/non-empty values of the 4-intersection *and* different dimensions of their  $\partial \cap \partial$  (for the sake of clarity, the interior 1-simplices has been omitted in the renderings): 0D-meet and 1D-meet, 0D-covers and 1D-covers—or reversely, 0D-coveredBy and 1D-coveredBy—and 0D-overlap and 1D-overlap, where 0D and 1D refers to 0-dimensional and 1-dimensional intersections, respectively.

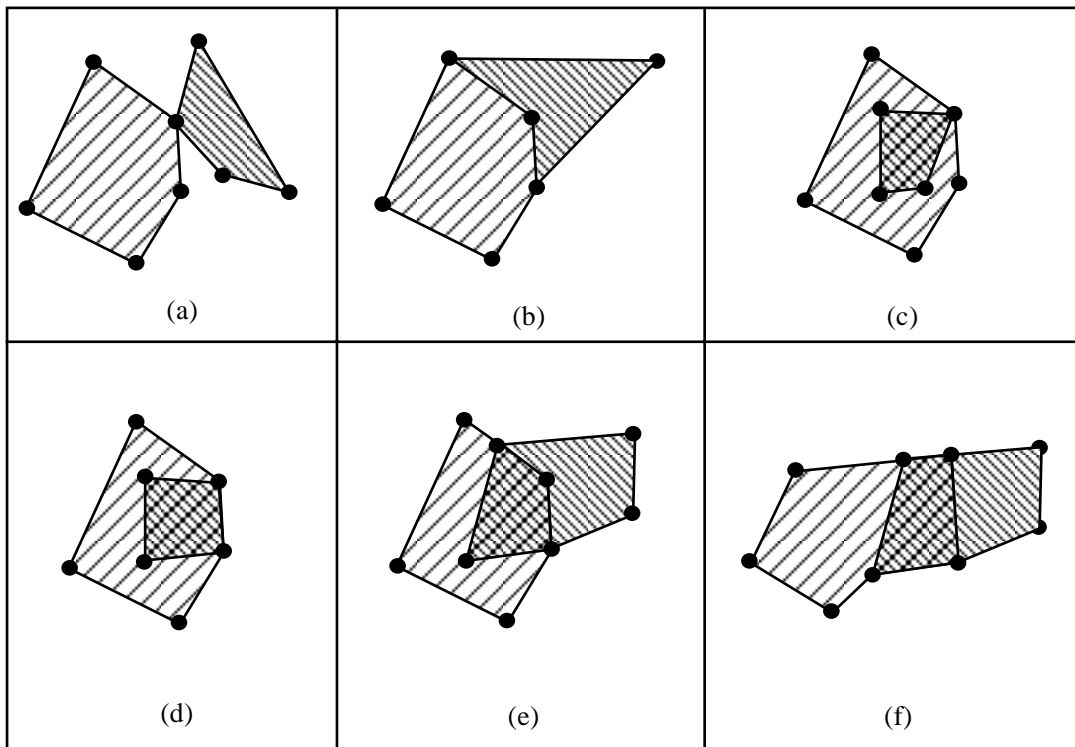


Figure 6: Examples of the detailed relationships between 2-complexes in  $\mathbb{R}^2$  considering different dimensions in the non-empty boundary intersections: (a) 0D-meet, (b) 1D-meet, (c) 0D-covers, (d) 1D-covers, (e) 0D-overlap, and (f) 1D-overlap.

## 4.2 Number of Separations

Another topological invariant to be investigated here is the *number of separations* (#) in the intersections. Again this will be considered a refinement of the empty/non-empty intersections, such that  $I_{\emptyset, \#}$  describes details about the non-empty intersections in terms of the number of separations. A separation of an intersection  $i$  is a pair  $a, b$  satisfying the three conditions that  $a \neq \emptyset$  and  $b \neq \emptyset$ ,  $a \cup b = i$ , and  $a \cap b = \emptyset$ .

The measure for the separation is the topological invariant of the *Euler-Poincaré characteristic*  $\chi$ , which is the alternating sum of the faces with the same dimension, i.e.,

$$\chi(c_n) = \sum_{r=0}^n (-1)^r \alpha_r(c_n) \quad (41)$$

where  $\alpha_r$  is the number of  $r$ -dimensional faces and  $c_n$  the  $n$ -complex whose Euler-Poincaré characteristic is determined.

The intersection of the boundaries forms an  $n-1$ -complex, because if an  $i$ -simplex is an element of both boundaries, then all its faces of dimension  $i-1$  through 0 are elements of their intersection as well. If the boundaries of both  $n$ -complexes coincide, their intersection is an  $n-1$ -sphere. If the boundaries coincide partially, the intersection can contain any subpart thereof; therefore, the Euler-Poincaré characteristic of the intersection of the boundaries is greater than or equal to 0 (Equation 42).

The interior forms an  $n-1$ -complex without any bounding faces; therefore, in codimension 0, the intersection of the interiors is an  $n-1$ -complex without any bounding faces as well. Its Euler-Poincaré characteristic depends upon the dimension of the complex—it is positive for even-dimensional complexes and negative for complexes with odd dimension—so that  $|\chi(\partial \cap \circ)|$  and  $|\chi(\circ \cap \partial)|$  are the numbers of separations (Equation 45). Likewise, the two intersections of boundary and interior form an  $n-1$ -complex without bounding faces, unless the boundary is a subset of the other object's interior (Equation 43 and 44).

$$\chi(\partial A_n \cap \partial B_n) \geq 0 \quad (42)$$

$$\chi(\partial A_n \cap B_n^\circ) = i \cdot (-1)^{n-1} \text{ with } i > 0 \quad (43)$$

$$\chi(B_n^\circ \cap \partial A_n) = i \cdot (-1)^{n-1} \text{ with } i > 0 \quad (44)$$

$$\chi(A_n^\circ \cap B_n^\circ) = i \cdot (-1)^n \text{ with } i > 0 \quad (45)$$

For example, the Euler-Poincaré characteristics for the four intersections (21–23) of the relationship between the two 2-complexes in Figure 5 are calculated as follows:

$$\chi(\partial A_2 \cap \partial B_2) = (-1)^0 \cdot 2s_0 + (-1)^1 \cdot 1s_1 = 1 \quad (46)$$

$$\chi(\partial A_2 \cap B_2^\circ) = (-1)^0 \cdot 1s_0 + (-1)^1 \cdot 2s_1 = -1 \quad (47)$$

$$\chi(A_2^\circ \cap \partial B_2) = 0 \quad (48)$$

$$\chi(A_2^\circ \cap B_2^\circ) = (-1)^0 \cdot 0s_0 + (-1)^1 \cdot 0s_1 + (-1)^2 \cdot 1 \cdot s_2 = 1 \quad (49)$$

The seven specifications with non-empty intersections (Equations 2–8) are candidates for detailed topological relationships due to their numbers of separated segments.  $R_{\text{equal}}$ ,  $R_{\text{inside}}$ , and  $R_{\text{contains}}$  are such that  $|\chi(I)|$  is constant (Equations 50–52) so that no separations in the non-empty intersections can be observed.

$$R_{\chi\text{-equal}}(A_n, B_n) \Leftrightarrow I_{\emptyset, \chi}(A_n, B_n) = \begin{pmatrix} -\emptyset & \emptyset \\ \chi=0 & \emptyset \\ \emptyset & -\emptyset \\ & \chi=(-1)^n \end{pmatrix} \equiv \begin{pmatrix} -\emptyset & \emptyset \\ \emptyset & -\emptyset \end{pmatrix} \quad (50)$$

$$R_{\chi\text{-}\bar{c}}(A_n, B_n) \Leftrightarrow I_{\emptyset, \chi}(A_n, B_n) = \begin{pmatrix} \emptyset & \emptyset \\ \neg\emptyset & \neg\emptyset \\ \chi=0 & \chi=(-1)^n \end{pmatrix} \equiv \begin{pmatrix} \emptyset & \emptyset \\ \neg\emptyset & \neg\emptyset \end{pmatrix} \quad (51)$$

$$R_{\chi\text{-}\bar{c}}(A_n, B_n) \Leftrightarrow I_{\emptyset, \chi}(A_n, B_n) = \begin{pmatrix} \emptyset & \neg\emptyset \\ \emptyset & \neg\emptyset \\ \chi=0 & \chi=(-1)^n \end{pmatrix} \equiv \begin{pmatrix} \emptyset & \neg\emptyset \\ \emptyset & \neg\emptyset \end{pmatrix} \quad (52)$$

The remaining four specifications with at least one non-empty intersection can have different Euler-Poincaré characteristics. The Euler-Poincaré characteristic of the non-empty  $\partial \cap \partial$  of  $R_{\text{meet}}$  is the number of separations in  $\partial \cap \partial$  and is always greater than 0 (Equation 53). Of the three non-empty specifications of  $R_{\text{covers}}$ ,  $|\chi(\circ \cap \circ)|$  is constant, while  $\chi(\partial \cap \partial)$  and  $|\chi(\circ \cap \partial)|$  depend upon each other and take the same value, i.e., the number of separated boundary segments is the same as the number of separated segments in  $\circ \cap \partial$  (Equation 54). The converse holds for  $R_{\text{coveredBy}}$  (Equation 55). Finally, the Euler-Poincaré characteristic of the four intersections of  $R_{\text{overlap}}$  can have a variety of values. If the interior-interior intersection has  $i$  separations, then the other three intersections have at least  $i + 1$  separations. There is no other dependency among the Euler-Poincaré characteristics of the other intersections (Equation 56).

$$R_{\chi\text{-meet}}(A_n, B_n) \Leftrightarrow I_{\emptyset, \chi}(A_n, B_n) = \begin{pmatrix} \neg\emptyset & \emptyset \\ \chi \geq i & \emptyset \\ \emptyset & \emptyset \end{pmatrix} \quad (53)$$

$$R_{\chi\text{-covers}}(A_n, B_n) \Leftrightarrow I_{\emptyset, \chi}(A_n, B_n) = \begin{pmatrix} \neg\emptyset & \emptyset \\ \chi=i & \emptyset \\ \neg\emptyset & \neg\emptyset \\ \chi=i \cdot (-1)^{n-1} & \chi=(-1)^n \end{pmatrix} \quad (54)$$

$$R_{\chi\text{-coveredBy}}(A_n, B_n) \Leftrightarrow I_{\emptyset, \chi}(A_n, B_n) = \begin{pmatrix} \neg\emptyset & \neg\emptyset \\ \chi=i & \chi=i \cdot (-1)^{n-1} \\ \emptyset & \neg\emptyset \\ \emptyset & \chi=(-1)^n \end{pmatrix} \quad (55)$$

$$R_{\chi\text{-overlap}}(A_n, B_n) \Leftrightarrow I_{\emptyset, \chi}(A_n, B_n) = \begin{pmatrix} \neg\emptyset & \neg\emptyset \\ \chi \geq i & \chi \geq i \\ \neg\emptyset & \neg\emptyset \\ \chi \geq i & \chi=i \cdot (-1)^n \end{pmatrix} \quad (56)$$

Figure 7 shows examples for those topological relationships between 2-complexes in  $\mathbb{R}^2$  that can be refined due to their Euler-Poincaré characteristic (again omitting the interior 1-simplices): #1-meet and #2-meet, #1-covers and #2-covers—or conversely, #1-coveredBy and #2-coveredBy—and #2-overlap and #4-overlap.

## 5 Conclusions

A formalism for the definition of topological relationships was presented, which is based upon purely topological properties and thus independent from the existence of a distance function or any other non-topological concepts. A generalized spatial data model, based upon simplicial complexes, was used to describe the fundamental operators of *set-theoretic boundary* and *interior*. The topological relationships were described by the intersections of boundary and interior with the binary values “empty” and “non-empty,” which distinguished eight topological relationships between two 2-dimensional simplicial complexes embedded in  $\mathbb{R}^2$ .

The major novel contribution was the distinction of further topological relationships as refinements of the empty/non-empty specifications based upon the recognition of other topological invariants. By considering the dimension of the intersections, detailed topological relationships were distinguished for *meet*, *covers*, *coveredBy*, and *overlap*. The same topological relationships could be refined by considering the number of separations in the intersections. Given these additional criteria, it is possible to detect which of the six configurations in the motivating example (Figure 1)

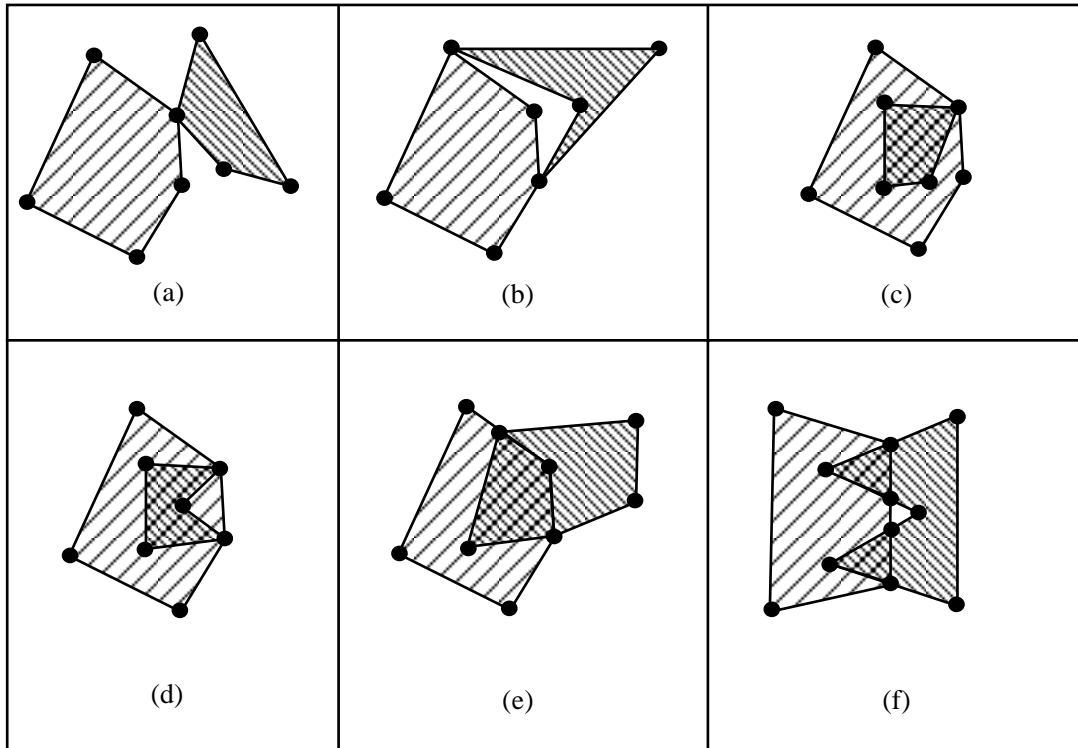


Figure 7: Examples of the detailed relationships between 2-complexes in  $\mathbb{R}^2$  considering different Euler-Poincare characteristics of the non-empty intersections: (a) #1-meet, (b) #2-meet, (c) #1-covers, (d) #2-covers, (e) #1-overlap, and (f) #2-overlap.

are topologically distinct (Table 2): (1a) is 0D-#2-overlap, (1b) 0D-#4-overlap, (1c) is 1D-#2-meet while (1d) is 0D-#1-meet, and (1e) and (1f) are both disjoint.

Table 2: The specifications of the six topological relationships in Figure 1.

$\begin{pmatrix} \neg\emptyset & \neg\emptyset \\ dim = 0 & \chi = -1 \\ \chi = 2 & \neg\emptyset \\ \neg\emptyset & \chi = 1 \end{pmatrix}$ $I_{\emptyset, dim, \chi}$ (Figure 1a)	$\begin{pmatrix} \neg\emptyset & \neg\emptyset \\ dim = 0 & \chi = -2 \\ \chi = 4 & \neg\emptyset \\ \neg\emptyset & \chi = 2 \end{pmatrix}$ $I_{\emptyset, dim, \chi}$ (Figure 1b)	$\begin{pmatrix} \neg\emptyset & \emptyset \\ dim = 1 & \emptyset \\ \chi = 2 & \emptyset \\ \emptyset & \emptyset \end{pmatrix}$ $I_{\emptyset, dim, \chi}$ (Figure 1c)
$\begin{pmatrix} \neg\emptyset & \emptyset \\ dim = 0 & \emptyset \\ \chi = 1 & \emptyset \\ \emptyset & \emptyset \end{pmatrix}$ $I_{\emptyset, dim, \chi}$ (Figure 1d)	$\begin{pmatrix} \emptyset & \emptyset \\ \emptyset & \emptyset \end{pmatrix}$ $I_{\emptyset, dim, \chi}$ (Figure 1e)	$\begin{pmatrix} \emptyset & \emptyset \\ \emptyset & \emptyset \end{pmatrix}$ $I_{\emptyset, dim, \chi}$ (Figure 1f)

Our results generated a number of further questions, which are being investigated. First, the topological invariants have certainly not been treated exhaustively. There may be other topological invariants that can be used to distinguish further details about topological relationships. Recently, for example, the sequence of segments of different dimensions in the boundary intersection—observed along the boundary of one of the objects—and the sequence by traversing the coboundaries of a

face in the boundary intersection have been identified as two further topological invariants that distinguish more details (Herring 1991). Investigations are underway to identify the smallest set of topological invariants necessary to describe a homeomorphism between any two topological relations (Franzosa and Egenhofer 1992).

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