Deriving the Composition of Binary Topological Relations*

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Abstract

A new formalism is presented to derive knowledge about the composition of two binary topological relations over a common object. The formalism is based on a topological data model and compares the nine empty and non-empty intersections of interiors, boundaries, and exteriors between two objects. Based upon the transitivity of set inclusion, the intersections of the composed topological relations are derived. These intersections are then matched with the intersections of the eight fundamental topological relations, giving an interpretation to the composition of topological relations. The result of this study is the composition table of the eight binary topological relations that exist between n-dimensional point sets with a codimension of 0. While the combined topological relations are unique for some compositions, more than half of all possible compositions are disjunctions of possible relations. Geometric prototypes are shown for the 2-dimensional case. The composition table enables topological reasoning at the conceptual level of relations, rather than having to calculate all relations from the representation of the spatial objects. Its practical value is that it can serve as in a computational model for an assessment of whether a set of topological predicates is consistent or not and in spatial query processing when no explicit information about spatial relations is available.

1 Introduction

Spatial reasoning has gained increasing popularity in recent years with applications in geographic information systems, navigation, robotics, computer vision, image understanding, pictorial databases, and CAD/CAM. The major factor contributing to the interest in spatial reasoning is that it offers its users new spatial information, which has not been explicitly recorded and which is otherwise not immediately available in the form of raw data. This paper focuses on *topological relations*, i.e., those *spatial relations* that are invariant under topological transformations and, therefore, preserved if the objects are translated, rotated, or scaled. Traditionally, reasoning about spatial relations has focused on formalisms to combine knowledge about *directions*, such as left, right, in front of, and

*This research was partially funded by NSF grant IRI-9309230 and grants from Intergraph Corporation. The support from NSF for the NCGIA under SES-880917 is gratefully acknowledged.

behind (6) or, in geographic applications, north, east, south, and west (7, 8, 16, 18, 26, 31). Frequently, the relations between 1-dimensional intervals (3) have been used as a basis for extensions to higher dimensions (17, 19, 21, 29, 33) and sometimes, topological properties have been inferred from non-topological concepts, e.g., from metric (32) or order (5, 27); however, such an approach disregards the fact that topological properties are most fundamental, compared to those of Euclidean, metric, and vector spaces, so that topological reasoning should be independent of these concepts (4). Since topology is a purely qualitative concept, independent of any quantitative measures, it has been difficult to find appropriate formal models for topological relations and methods to combine topological knowledge and reason about them (20). Unlike any pictorial representation of topological relations, which inevitably combines topological information with such non-topological information as relative distances, directions, sizes, orientations, and shapes of the objects, a propositional representation of topological relations allows for an exclusive focus on topological properties (24, 36).

This work is a continuation of our efforts to formalize spatial relations as they are used in geographic information systems (1, 28). Previous results included a framework for analyzing topological relations (13, 12) and a categorization of all binary topological relations between all combinations of points, lines, and regions (14). This paper develops a new formalism to *integrate* topological information and infer knowledge about the composition of binary topological relations to answer questions of the type, "Given three objects, A, B, and C and the two topological relations A r_i B and B r_j C, what is the topological relation A r_k C?" With our method, the complete set of relations from the compositions of two binary topological relations for 2-dimensional objects in $\mathbb{R}^2$ has been determined. This can be compared to the analogous composition of similar relations between 1-dimensional intervals, which was called the *transitivity table* (3).

The composition of topological relations is an essential part of a *relation algebra* (29, 37) for spatial relations. Spatial databases will benefit from the composition table of topological relations if it is applied during data acquisition to integrate independently collected topological information and to derive new topological knowledge; to detect consistency violations among spatial data about some otherwise non-evident topological facts (15); or during query processing, when spatial queries with complex topological constraints can be substituted by simpler operations, which are either less expensive to be executed or involve less objects.

The remainder of this paper is structured as follows: Section 2 summarizes our model for binary topological relations based upon the nine intersections of interiors, boundaries, and exteriors. In Section 3 the composition of two binary topological relations is derived from the transitive property of subsets applied to the 9-intersection representation for interior, boundary, and exterior intersections. Two examples of this inference process are given in Section 4 and the set of all possible compositions of the eight topological relations between two 2-dimensional point sets is shown in Section 5. Section 6 presents the conclusions and discusses the application of this method.

2 Topological Relations

The usual concepts of point-set topology with open and closed sets are assumed (2, 35). The *interior* of a set A, denoted by A°, is the union of all open sets in A. The *closure* of A, denoted by $\overline{A}$, is the intersection of all closed sets of A. The *complement* of A with respect to the embedding space $\mathbb{R}^2$ (or the *exterior*, denoted by A^-, is the set of all points of $\mathbb{R}^2$ not contained in A. The *boundary* of A, denoted by ∂A, is the intersection of the closure of A and the closure of the exterior of A. ∂A, A°, and A^- are mutually exclusive and $\partial A \cup A^\circ \cup A^-$ is $\mathbb{R}^2$.

Subsequently, interior,boundary, and exterior will be sometimes referred to as the three *object parts*. The topological relation r_n between two point sets, A and B, is described by the nine set intersections I of A's interior, boundary, and exterior with the interior, boundary, and exterior of B, called the *9-intersection* (14). Indices like I_x and I_y will be used whenever it is necessary to distinguish between different 9-intersections.

$$I = \begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^- \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap B^\circ & A^- \cap \partial B & A^- \cap B^- \end{pmatrix}$$

Let the parts a and b be arbitrary elements of $\{A^\circ, \partial A, A^-\}$ and $\{B^\circ, \partial B, B^-\}$, respectively. An index, like a_i and b_i, will indicate corresponding elements of the two sets A and B, i.e., either both interiors, both boundaries, or both exteriors. The notion $I[a_i, b_j]$ will be used to refer to a particular intersection. For instance, $I[\partial,^\circ]$ would describe the boundary-interior intersection, $\partial A \cap B^\circ$.

The 9-intersection is an extended representation of the initially proposed 4-intersection consisting of the four set intersections of interiors and boundaries (9, 13). It is superior over the 4-intersection, because it also considers—besides the relationships among the object parts—their relationships with respect to the embedding space. This extension is necessary to determine whether or not an intersection is *completely* included in an object part (14).

Various topological invariants can be used to evaluate I and characterize the topological relation $A \ \mathrm{r}_n \ B$. Most fundamental is the distinction of the values empty ($\emptyset$) and non-empty ($\neg\emptyset$), which gives rise to 2^9 possible combinations (12, 13). They provide a complete coverage and are mutually exclusive, so that for every possible configuration between two objects, always exactly one empty/non-empty 9-intersection exists. Previous investigations found that only a small subset of the 512 possible relations can be realized. In $\mathbb{R}^2$, there are 8 relations between two spatial regions without holes (2-dimensional, connected objects with connected boundaries) (12); 18 between spatial regions with holes; 33 between two simple lines; and 19 between a spatial region without holes and a simple line (14). Fig. 1 shows the eight specifications of the binary topological relations that can be realized between two spatial regions without holes and prototypes of the geometric interpretations of the corresponding relations. These eight relations between two spatial regions without holes will be the exclusive focus in the examples of this paper; however, the concepts developed will be applicable to any other pair of topological relations modeled by the 9-intersection.

Combinatorial topology (30) shifts the focus from the infinite point set, which cannot be directly represented in a computer, to the finite set of points, lines, and areas. The topological concepts apply directly and can be formalized as an algebra over chains, either for simplicial complexes (11, 39) or cells (22). The concept of the intersections between interiors and boundaries has been mapped from point sets onto such a spatial data models using algebraic topology (9) including algorithms for the efficient calculation of interiors, boundaries, and their intersections for objects represented by such a topological data structure (14). A variation of the 9-intersection has been successfully implemented in the commercial geographic information system MGE/Dynamo (23).

3 Composition of Topological Relations

Two binary topological relations can be combined if both relations share a common object. This problem is somehow similar to determining whether or not a relation is transitive and sometimes even referred to as the *transitive relation* (3); however, unlike a transitive relation, which combines the same two relations, the composition of relations links two potentially different relations (29). We use the operator ; to denote the composition (37). For example, A inside B and B inside $C \Rightarrow A$ inside C will be simply expressed as inside ; inside $\Rightarrow$ inside.

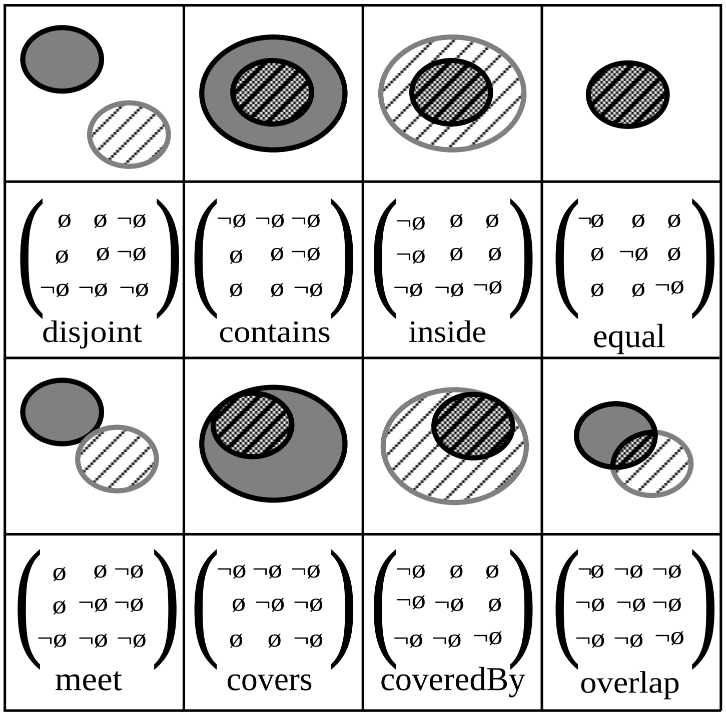

Figure 1. Examples of the relations between two spatial regions without holes in $\mathbb{R}^2$.

3.1 Set Inclusion and Containment of the 9-Intersection

Given that the topological relations are represented by the 9-intersection, their composition can be determined by deriving the 9-intersection of the combined topological relation. Since spatial knowledge about topological relations is encoded into set intersections, standard transitive inference rules about point sets can be applied. Every object has exactly three parts (interior, boundary, and exterior), which can be in a particular relationship to the parts of another object, and vice-versa. Together, there are eight relevant combinations that derive knowledge about the combined relation. They are statements of implications that can be proven to be theorems of set theory:

Theorem 1 *A non-empty intersection between two parts A and B implies a non-empty intersection between the parts A and C if B is a subset of C, i.e.,*

$$A \cap B = \neg\emptyset \quad \wedge \quad B \subseteq C \quad \Rightarrow \quad A \cap C = \neg\emptyset \tag{1}$$

Proof: Let x be a non-empty element such that $x \in A$ and $x \in B$. Since $B \subseteq C$, $x \in C$ as well. Thus, $x \in (A \cap C)$ and, therefore, $A \cap C = \neg\emptyset$. □

Corollary 2

$$A \supseteq B \quad \wedge \quad B \cap C = \neg\emptyset \quad \Rightarrow \quad A \cap C = \neg\emptyset \tag{2}$$

Proof: By replacing A and C in (1) with C and A, respectively, and reordering the terms on the left-hand side. □

Theorem 3 *An empty intersection between the parts A and B implies an empty intersection between the parts A and C if C is a subset of B, i.e.,*

$$A \cap B = \emptyset \quad \wedge \quad B \supseteq C \quad \Rightarrow \quad A \cap C = \emptyset \tag{3}$$

Proof: Let $x \in C$. Since $B \supseteq C$, $x \in B$ as well. On the other hand, since $A \cap B = \emptyset$, $x \in B$ implies $x \notin A$. Therefore, $x \in C$ implies $x \notin A$, or $C \cap A = \emptyset$. □

Corollary 4

$$A \subseteq B \quad \wedge \quad B \cap C = \emptyset \quad \Rightarrow \quad A \cap C = \emptyset \tag{4}$$

Proof: By replacing A and C in (4) with C and A, respectively, and reordering the terms on the left-hand side. □

Corollary 5 *A non-empty intersection between the parts A and B implies a non-empty intersection with the union of the two parts C_0 and C_1 if B is a subset of the union of C_0 and C_1, i.e.,*

$$A \cap B = \neg\emptyset \quad \wedge \quad B \subseteq (C_0 \cup C_1) \quad \Rightarrow \quad A \cap (C_0 \cup C_1) = \neg\emptyset \tag{5}$$

Proof: Immediately from Theorem 1, substituting C with $(C_0 \cup C_1)$. □

Corollary 6

$$(A_0 \cup A_1) \supseteq B \quad \wedge \quad B \cap C = \neg\emptyset \quad \Rightarrow \quad (A_0 \cup A_1) \cap C = \neg\emptyset \tag{6}$$

Proof: By replacing A, C_0, and C_1 in (5) with C, A_0, and A_1, respectively, and reordering the terms on the left-hand side. □

Corollary 7 *An empty intersection between A and the union of B_0 and B_1 implies an empty intersection between A and C if the C is a subset of the union of B_0 and B_1, i.e.,*

$$A \cap (B_0 \cup B_1) = \emptyset \quad \wedge \quad (B_0 \cup B_1) \supseteq C \quad \Rightarrow \quad A \cap C = \emptyset \tag{7}$$

Proof: Immediately from theorem 3, substituting B with $(B_0 \cup B_1)$. □

Corollary 8

$$A \subseteq (B_0 \cup B_1) \quad \wedge \quad (B_0 \cup B_1) \cap C = \emptyset \quad \Rightarrow \quad A \cap C = \emptyset \tag{8}$$

Proof: By replacing A and C in (7) with C and A, respectively, and reordering the terms on the left-hand side. □

This set of eight rules is sufficiently complete to describe the dependencies of the intersections. Further considerations about the union of *three* parts are unnecessary since these cases are trivial. For instance, the derived intersections of non-empty intersections over the union of three parts are impossible since every part must be included in the universe. On the other hand, the following constraint must hold true for every non-empty intersection, because it is impossible that all three intersections with another part are empty:

$$A \cap B = \neg\emptyset \quad \wedge \quad B \subseteq (C_0 \cup C_1 \cup C_2) \quad \Rightarrow \quad A \cap (C_0 \cup C_1 \cup C_2) = \neg\emptyset \tag{9}$$

The eight intersections are not orthogonal since Eqs. (1–4) are included in Eqs. (5–8), respectively, if A is a subset of B_0 or B_1. This redundancy is eliminated if Eqs. (5–8) are modified so that they exclude the configurations covered by Eqs. (1–4). Let $\sqsubseteq$ be the relationship between a set A and the union of the sets B and C such that $A \subseteq (B \cup C)$ and $A \cap B = \neg\emptyset$ and $A \cap C = \neg\emptyset$ (the relationship $\sqsupseteq$ can be defined correspondingly).

$$A \cap B = \neg\emptyset \quad \wedge \quad B \sqsubseteq (C_0 \cup C_1) \quad \Rightarrow \quad A \cap (C_0 \cup C_1) = \neg\emptyset \tag{10}$$

$$(A_0 \cup A_1) \sqsupseteq B \quad \wedge \quad B \cap C = \neg\emptyset \quad \Rightarrow \quad (A_0 \cup A_1) \cap C = \neg\emptyset \tag{11}$$

$$A \cap (B_0 \cup B_1) = \emptyset \quad \wedge \quad (B_0 \cup B_1) \sqsupseteq C \quad \Rightarrow \quad A \cap C = \emptyset \tag{12}$$

$$A \sqsubseteq (B_0 \cup B_1) \quad \wedge \quad (B_0 \cup B_1) \cap C = \emptyset \quad \Rightarrow \quad A \cap C = \emptyset \tag{13}$$

3.2 Transforming Subsets into the 9-Intersection

The eight rules Eqs. (1–4) and (10–13) can be applied to drive the 9-intersection of the combined topological relation if $A \subseteq B$, $A \supseteq B$, $A \sqsubseteq (B_0 \cup B_1)$, $(A_0 \cup A_1) \sqsupseteq B$, $A \cap (B_0 \cup B_1) = \neg\emptyset$, and $A \cap (B_0 \cup B_1) = \emptyset$ can be represented in terms of the 9-intersection ($A \cap B = \emptyset$ and $A \cap B = \neg\emptyset$ are already in this canonical representation). The following transformations apply: Let $a_i \neq a_j \neq a_k$, $b_l \neq b_m \neq b_n$, and $c_o \neq c_p \neq c_q$.

a_i is a subset of b_l if and only if $I[a_i, b_l]$ is non-empty, while the two intersections between a_i and the other two parts b_m and b_n are empty (Eq. 14). This mapping of the subset relation onto the 9-intersection is obvious, because the non-empty intersection between a_i and b_l is immediately derived from the subset relation between non-empty sets. Since the three parts of B are pairwise disjoint, a non-empty intersection $a_i \cap b_m$ or $a_i \cap b_n$ would imply that there are some parts of a_i outside of b_l, which would contradict the subset relation (recall that $I[a_x, b_y]$ denotes an intersection of the interiors, boundaries, and exteriors of the two objects A and B).

$$I[a_i, b_l] = \neg\emptyset \quad \wedge \quad I[a_i, b_m] = \emptyset \quad \wedge \quad I[a_i, b_n] = \emptyset \tag{14}$$

Conversely, a_i is a superset of b_l if and only if

$$I[a_i, b_l] = \neg\emptyset \quad \wedge \quad I[a_j, b_l] = \emptyset \quad \wedge \quad I[a_k, b_l] = \emptyset \tag{15}$$

$a_i \sqsubseteq (b_l \cup b_m)$ if the intersections $I[a_i, b_l]$ and $I[a_i, b_m]$ are non-empty, while the third intersection between a_i and b_n is empty (Eq. 16). By the definition of $\sqsubseteq$, $a_i \cap b_l = \neg\emptyset$ and $a_i \cap b_m = \neg\emptyset$. Since b_n is disjoint from both b_l and b_m, its intersection with a_i must be empty, otherwise a_i would have some parts outside of $b_l \cup b_m$, which would contradict the subset relation.

$$I[a_i, b_l] = \neg\emptyset \quad \wedge \quad I[a_i, b_m] = \neg\emptyset \quad \wedge \quad I[a_i, b_n] = \emptyset \tag{16}$$

Conversely, $(a_i \cup a_j) \sqsupseteq b_l$ if

$$I[a_i, b_l] = \neg\emptyset \quad \wedge \quad I[a_j, b_l] = \neg\emptyset \quad \wedge \quad I[a_k, b_l] = \emptyset \tag{17}$$

The intersection of a_i with $b_l \cup b_m$ is non-empty if at least one of the two intersections $I[a_i, b_l]$ and $I[a_i, b_m]$ is non-empty (Eq. 18).

$$\neg(I[a_i, b_l] = \emptyset \quad \wedge \quad I[a_i, b_m] = \emptyset) \tag{18}$$

Complementarily, $a_i \cap (b_l \cup b_m)$ is empty if and only if

$$I[a_i, b_l] = \emptyset \quad \wedge \quad I[a_i, b_m] = \emptyset \tag{19}$$

3.3 Inference Rules for the 9-Intersection

The eight inference rules about the intersections of the combined topological relations are derived by using Eqs. (14–19) in (1–4) and (10–13).

(Eq. 14) in (Eq. 1) :
$$I_x[a_i, b_l] = \neg\emptyset \\ \wedge \; I_y[b_l, c_o] = \emptyset \quad \wedge \quad I_y[b_l, c_p] = \emptyset \quad \wedge \quad I_y[b_l, c_q] = \neg\emptyset \\ \Rightarrow \quad I_z[a_i, c_q] = \neg\emptyset \tag{20}$$

(Eq. 15) in (Eq. 2) :
$$I_x[a_i, b_l] = \neg\emptyset \quad \wedge \quad I_x[a_j, b_l] = \emptyset \quad \wedge \quad I_x[a_k, b_l] = \emptyset \\ \wedge \; I_y[b_l, c_o] = \neg\emptyset \\ \Rightarrow \quad I_z[a_i, c_o] = \neg\emptyset \tag{21}$$

(Eq. 15) in (Eq. 3) :
$$I_x[a_i, b_l] = \emptyset \\ \wedge \; I_y[b_l, c_o] = \neg\emptyset \quad \wedge \quad I_y[b_m, c_o] = \emptyset \quad \wedge \quad I_y[b_n, c_o] = \emptyset \\ \Rightarrow \quad I_z[a_i, c_o] = \emptyset \tag{22}$$

(Eq. 14) in (Eq. 4) :
$$I_x[a_i, b_l] = \neg\emptyset \quad \wedge \quad I_x[a_i, b_m] = \emptyset \quad \wedge \quad I_x[a_i, b_n] = \emptyset \\ \wedge \; I_y[b_l, c_o] = \emptyset \\ \Rightarrow \quad I_z[a_i, c_o] = \emptyset \tag{23}$$

(Eq. 16) and (Eq. 18) in (Eq. 10) :
$$I_x[a_i, b_l] = \neg\emptyset \\ \wedge \; I_y[b_l, c_o] = \neg\emptyset \quad \wedge \quad I_y[b_l, c_p] = \emptyset \quad \wedge \quad I_y[b_l, c_q] = \neg\emptyset \\ \Rightarrow \quad \neg(I_z[a_i, c_o] = \emptyset \quad \wedge \quad I_z[a_i, c_q] = \emptyset) \tag{24}$$

(Eq. 17) and (Eq. 18) in (Eq. 11) :
$$I_x[a_i, b_l] = \neg\emptyset \quad \wedge \quad I_x[a_j, b_l] = \neg\emptyset \quad \wedge \quad I_x[a_k, b_l] = \emptyset \\ \wedge \; I_y[b_l, c_o] = \neg\emptyset \\ \Rightarrow \quad \neg(I_z[a_i, c_o] = \emptyset \quad \wedge \quad I_z[a_j, c_o] = \emptyset) \tag{25}$$

(Eq. 19) and (Eq. 17) in (Eq. 12) :
$$I_x[a_i, b_l] = \emptyset \quad \wedge \quad I_x[a_i, b_m] = \emptyset \\ \wedge \; I_y[b_l, c_o] = \neg\emptyset \quad \wedge \quad I_y[b_m, c_o] = \neg\emptyset \quad \wedge \quad I_y[b_n, c_o] = \emptyset \\ \Rightarrow \quad I_z[a_i, c_o] = \emptyset \tag{26}$$

(Eq. 16) and (Eq. 19) in (Eq. 13) :
$$I_x[a_i, b_l] = \neg\emptyset \quad \wedge \quad I_x[a_i, b_m] = \emptyset \quad \wedge \quad I_x[a_i, b_n] = \neg\emptyset \\ \wedge \; I_y[b_l, c_o] = \emptyset \quad \wedge \quad I_y[b_n, c_o] = \emptyset \\ \Rightarrow \quad I_z[a_i, c_o] = \emptyset \tag{27}$$

3.4 Integrating Inferred Intersections

The intersections of the combined topological relations are described by the combination of the results of Eqs. (23–27). The following must be considered when these intersections are integrated:

- Some intersections may be multiply derived, because the inference rules, Eqs. (23–27), determine each individual intersection over several paths; therefore, only those combined topological relations are valid which match with at least one of the eight intersections. A contradiction among two or more redundantly derived values exists if $\emptyset$ and $\neg\emptyset$ would be derived as the values of an intersection. This would indicate that the combined topological relation does not exist.

- The union of all compositions may be insufficient to identify a *unique* combined relation. In such ambiguous cases, the result is a set of *possible* relations which comprises all those relations whose intersections do not contradict the derived values. Since the set of topological relations is finite, the result can be also transformed such that it describes the complement, i.e., the set of *impossible* relations.

4 Examples

This section will give two examples of determining the composition of two topological relations for two specific cases. The first derives a unique topological relation and is to focus the reader's attention on the process of deriving and combining intersections. The second example primarily demonstrates how imprecise information is derived.

4.1 Meet ; contains

The intersections of the composition of the topological relations meet and contains are determined as follows (their 9-intersections are taken from Fig. 1 such that meet corresponds to I_1 and contains to I_5):

$$I_1[\partial,\partial], I_5[\partial,\_] \text{ in (Eq. 20)}: \quad \begin{aligned} & I_1[\partial,\partial] = \neg\emptyset \\ & \wedge\ I_5[\partial,\partial] = \emptyset \ \wedge\ I_5[\partial,^\circ] = \emptyset \ \wedge\ I_5[\partial,^-] = \neg\emptyset \\ & \Rightarrow\ I_{1;5}[\partial,^-] = \neg\emptyset \end{aligned} \tag{28}$$

$$I_1[\partial,^-], I_5[^-,\_] \text{ in (Eq. 20)}: \quad \begin{aligned} & I_1[\partial,^-] = \neg\emptyset \\ & \wedge\ I_5[^-,\partial] = \emptyset \ \wedge\ I_5[^-,^\circ] = \emptyset \ \wedge\ I_5[^-,^-] = \neg\emptyset \\ & \Rightarrow\ I_{1;5}[\partial,^-] = \neg\emptyset \end{aligned} \tag{29}$$

$$I_1[^\circ,^-], I_5[^-,\_] \text{ in (Eq. 20)}: \quad \begin{aligned} & I_1[^\circ,^-] = \neg\emptyset \\ & \wedge\ I_5[^-,\partial] = \emptyset \ \wedge\ I_5[^-,^\circ] = \emptyset \ \wedge\ I_5[^-,^-] = \neg\emptyset \\ & \Rightarrow\ I_{1;5}[^\circ,^-] = \neg\emptyset \end{aligned} \tag{30}$$

$$I_1[^-,\partial], I_5[\partial,\_] \text{ in (Eq. 20)}: \quad \begin{aligned} & I_1[^-,\partial] = \neg\emptyset \\ & \wedge\ I_5[\partial,\partial] = \emptyset \ \wedge\ I_5[\partial,^\circ] = \emptyset \ \wedge\ I_5[\partial,^-] = \neg\emptyset \\ & \Rightarrow\ I_{1;5}[^-,^-] = \neg\emptyset \end{aligned} \tag{31}$$

$$I_1[^-,^-], I_5[^-,\_] \text{ in (Eq. 20)}: \quad \begin{aligned} & I_1[^-,^-] = \neg\emptyset \\ & \wedge\ I_5[^-,\partial] = \emptyset \ \wedge\ I_5[^-,^\circ] = \emptyset \ \wedge\ I_5[^-,^-] = \neg\emptyset \\ & \Rightarrow\ I_{1;5}[^-,^-] = \neg\emptyset \end{aligned} \tag{32}$$

$$I_1[\_,^\circ], I_5[^\circ,\partial] \text{ in (Eq. 21)}: \quad \begin{aligned} & I_1[\partial,^\circ] = \emptyset \ \wedge\ I_1[^\circ,^\circ] = \emptyset \ \wedge\ I_1[^-,^\circ] = \neg\emptyset \\ & \wedge\ I_5[^\circ,\partial] = \neg\emptyset \\ & \Rightarrow\ I_{1;5}[^-,\partial] = \neg\emptyset \end{aligned} \tag{33}$$

$$I_1[_-,^\circ], I_5[^\circ,^\circ] \text{ in (Eq. 21)}: \quad I_1[\partial,^\circ] = \emptyset \;\wedge\; I_1[^\circ,^\circ] = \emptyset \;\wedge\; I_1[^-,^\circ] = \neg\emptyset \;\wedge\; I_5[^\circ,^\circ] = \neg\emptyset \;\Rightarrow\; I_{1;5}[^-,^\circ] = \neg\emptyset \tag{34}$$

$$I_1[_-,^\circ], I_5[^\circ,^-] \text{ in (Eq. 21)}: \quad I_1[\partial,^\circ] = \emptyset \;\wedge\; I_1[^\circ,^\circ] = \emptyset \;\wedge\; I_1[^-,^\circ] = \neg\emptyset \;\wedge\; I_5[^\circ,^-] = \neg\emptyset \;\Rightarrow\; I_{1;5}[^-,^-] = \neg\emptyset \tag{35}$$

$$I_1[^\circ,_-], I_5[^-,\partial] \text{ in (Eq. 23)}: \quad I_1[^\circ,\partial] = \emptyset \;\wedge\; I_1[^\circ,^\circ] = \emptyset \;\wedge\; I_1[^\circ,^-] = \neg\emptyset \;\wedge\; I_5[^-,\partial] = \emptyset \;\Rightarrow\; I_{1;5}[^\circ,\partial] = \emptyset \tag{36}$$

$$I_1[^\circ,_-], I_5[^-,^\circ] \text{ in (Eq. 23)}: \quad I_1[^\circ,\partial] = \emptyset \;\wedge\; I_1[^\circ,^\circ] = \emptyset \;\wedge\; I_1[^\circ,^-] = \neg\emptyset \;\wedge\; I_5[^-,^\circ] = \emptyset \;\Rightarrow\; I_{1;5}[^\circ,^\circ] = \emptyset \tag{37}$$

$$I_1[\partial,^\circ], I_5[_-,\partial] \text{ in (Eq. 22)}: \quad I_1[\partial,^\circ] = \emptyset \;\wedge\; I_5[\partial,\partial] = \emptyset \;\wedge\; I_5[^\circ,\partial] = \neg\emptyset \;\wedge\; I_5[^-,\partial] = \emptyset \;\Rightarrow\; I_{1;5}[\partial,\partial] = \emptyset \tag{38}$$

$$I_1[^\circ,^\circ], I_5[_-,\partial] \text{ in (Eq. 22)}: \quad I_1[^\circ,^\circ] = \emptyset \;\wedge\; I_5[\partial,\partial] = \emptyset \;\wedge\; I_5[^\circ,\partial] = \neg\emptyset \;\wedge\; I_5[^-,\partial] = \emptyset \;\Rightarrow\; I_{1;5}[^\circ,\partial] = \emptyset \tag{39}$$

$$I_1[\partial,^\circ], I_5[_-,^\circ] \text{ in (Eq. 22)}: \quad I_1[\partial,^\circ] = \emptyset \;\wedge\; I_5[\partial,^\circ] = \emptyset \;\wedge\; I_5[^\circ,^\circ] = \neg\emptyset \;\wedge\; I_5[^-,^\circ] = \emptyset \;\Rightarrow\; I_{1;5}[\partial,^\circ] = \emptyset \tag{40}$$

$$I_1[^\circ,^\circ], I_5[_-,^\circ] \text{ in (Eq. 22)}: \quad I_1[^\circ,^\circ] = \emptyset \;\wedge\; I_5[\partial,^\circ] = \emptyset \;\wedge\; I_5[^\circ,^\circ] = \neg\emptyset \;\wedge\; I_5[^-,^\circ] = \emptyset \;\Rightarrow\; I_{1;5}[^\circ,^\circ] = \emptyset \tag{41}$$

$$I_1[_-,\partial], I_5[\partial,^-] \text{ in (Eq. 25)}: \quad I_1[\partial,\partial] = \neg\emptyset \;\wedge\; I_1[^\circ,\partial] = \emptyset \;\wedge\; I_1[^-,\partial] = \neg\emptyset \;\wedge\; I_5[\partial,^-] = \neg\emptyset \;\Rightarrow\; \neg(I_{1;5}[\partial,^-] = \emptyset \;\wedge\; I_{1;5}[^-,^-] = \emptyset]) \tag{42}$$

$$I_1[\partial,_-], I_5[_-,\partial] \text{ in (Eq. 27)}: \quad I_1[\partial,\partial] = \neg\emptyset \;\wedge\; I_1[\partial,^\circ] = \emptyset \;\wedge\; I_1[\partial,^-] = \neg\emptyset \;\wedge\; I_5[\partial,\partial] = \emptyset \;\wedge\; I_5[^-,\partial] = \emptyset \;\Rightarrow\; I_{1;5}[\partial,\partial] = \emptyset \tag{43}$$

$$I_1[\partial,_-], I_5[_-,\partial] \text{ in (Eq. 27)}: \quad I_1[\partial,\partial] = \neg\emptyset \;\wedge\; I_1[\partial,^\circ] = \emptyset \;\wedge\; I_1[\partial,^-] = \neg\emptyset \;\wedge\; I_5[\partial,^\circ] = \emptyset \;\wedge\; I_5[^-,^\circ] = \emptyset \;\Rightarrow\; I_{1;5}[\partial,^\circ] = \emptyset \tag{44}$$

The calculation of $I_{1;5}[a,c]$ yields redundant specifications. For instance, $I_{1;5}[^-,^-]$ is determined three times. These redundancies are easily eliminated since they do not result in contradicting values for the intersections. The compilation of Eqs. (28–44) shows that all nine intersections $I_{1;5}[a,c]$ are determined (Eq. 45).

$$I_{1;5} = \begin{pmatrix} \emptyset & \emptyset & \neg\emptyset \\ \emptyset & \emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix} \tag{45}$$

The nine intersections are the same as the ones of I_0, therefore, the combined topological relation is

$$\text{meet ; contains} \quad \Rightarrow \quad \text{disjoint}$$

4.2 Meet ; coveredBy

The intersections of the composition of the topological relation meet and coveredBy are determined as follows (meet corresponds to I_1 and coveredBy to I_4):

$$\begin{aligned}
I_1[^-,^\circ], I_4[^\circ,_] \text{ in (Eq. 20)}: \quad & I_1[^-,^\circ] = \neg\emptyset \\
& \wedge \;\; I_4[^\circ,\partial] = \emptyset \;\;\wedge\;\; I_4[^\circ,^\circ] = \neg\emptyset \;\;\wedge\;\; I_4[^\circ,^-] = \emptyset \\
& \Rightarrow \;\; I_{1;4}[^-,^\circ] = \neg\emptyset \qquad (46) \\
I_1[_,^\circ], I_4[^\circ,^\circ] \text{ in (Eq. 21)}: \quad & I_1[\partial,^\circ] = \emptyset \;\;\wedge\;\; I_1[^\circ,^\circ] = \emptyset \;\;\wedge\;\; I_1[^-,^\circ] = \neg\emptyset \\
& \wedge \;\; I_4[^\circ,^\circ] = \neg\emptyset \\
& \Rightarrow \;\; I_{1;4}[^-,^\circ] = \neg\emptyset \qquad (47) \\
I_1[^-,\partial], I_4[\partial,_] \text{ in (Eq. 24)}: \quad & I_1[^-,\partial] = \neg\emptyset \\
& \wedge \;\; I_4[\partial,\partial] = \neg\emptyset \;\;\wedge\;\; I_4[\partial,^\circ] = \neg\emptyset \;\;\wedge\;\; I_4[\partial,^-] = \emptyset \\
& \Rightarrow \;\; \neg(I_{1;4}[^-,\partial] = \emptyset \;\;\wedge\;\; I_{1;4}[^-,^\circ] = \emptyset) \qquad (48) \\
I_1[_,\partial], I_4[\partial,\partial] \text{ in (Eq. 25)}: \quad & I_1[\partial,\partial] = \neg\emptyset \;\;\wedge\;\; I_1[^\circ,\partial] = \emptyset \;\;\wedge\;\; I_1[^-,\partial] = \neg\emptyset \\
& \wedge \;\; I_4[\partial,\partial] = \neg\emptyset \\
& \Rightarrow \;\; \neg(I_{1;4}[\partial,\partial] = \emptyset \;\;\wedge\;\; I_{1;4}[^-,\partial] = \emptyset) \qquad (49) \\
I_1[_,\partial], I_4[\partial,^\circ] \text{ in (Eq. 25)}: \quad & I_1[\partial,\partial] = \neg\emptyset \;\;\wedge\;\; I_1[^\circ,\partial] = \emptyset \;\;\wedge\;\; I_1[^-,\partial] = \neg\emptyset \\
& \wedge \;\; I_4[\partial,^\circ] = \neg\emptyset \\
& \Rightarrow \;\; \neg(I_{1;4}[\partial,^\circ] = \emptyset \;\;\wedge\;\; I_{1;4}[^-,^\circ] = \emptyset) \qquad (50) \\
I_1[_,\partial], I_4[\partial,^-] \text{ in (Eq. 25)}: \quad & I_1[\partial,\partial] = \neg\emptyset \\
& \wedge \;\; I_4[\partial,\partial] = \neg\emptyset \;\;\wedge\;\; I_4[\partial,^\circ] = \neg\emptyset \;\;\wedge\;\; I_4[\partial,^-] = \emptyset \\
& \Rightarrow \;\; \neg(I_{1;4}[\partial,\partial] = \emptyset \;\;\wedge\;\; I_{1;4}[\partial,^\circ] = \emptyset) \qquad (51)
\end{aligned}$$

The intersections $I_{1;4}[a,c]$ are compiled as the conjunction of positive and negative constraints for possible and impossible intersections. The summary of intersections, Eqs. (46–51), shows the following properties:

- $[^-,^\circ]$ is overdetermined, because Eq. (46) and Eq. (47) derive the same value for the same intersection.
- Eq. (48) is covered by Eq. (46), because $[^-,^\circ] = \neg\emptyset \;\;\wedge\;\; \neg([^-,^\circ] = \emptyset \wedge [^-,\partial] = \emptyset) \;\;=\;\; [^-,^\circ] = \neg\emptyset$. Likewise, Eq. (46) covers Eq. (50).
- $[\partial,\partial]$, $[\partial,^\circ]$, $[^-,\partial]$ are partially determined (Eqs. 49 and 51).
- $[\partial,^-]$, $[^-,^-]$, $[^\circ,\partial]$, $[^\circ,^\circ]$, $[^\circ,^-]$ are undetermined.

Therefore, Eqs. (46–51) can be represented by a sequence of disjunctions of 9-intersections with multiple undefined (_) specifications (Eq. 52).

$$I_{1;4} = \begin{pmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \neg\emptyset & \_ & \_ \end{pmatrix} \wedge \neg \begin{pmatrix} \_ & \_ & \_ \\ \_ & \emptyset & \_ \\ \_ & \emptyset & \_ \end{pmatrix} \wedge \neg \begin{pmatrix} \_ & \_ & \_ \\ \emptyset & \emptyset & \_ \\ \_ & \_ & \_ \end{pmatrix} \quad (52)$$

The comparison with the eight intersections that can be realized in $\mathbb{R}^2$ (Fig. 1) reveals that I_0 violates the condition in Eq. (51), I_2 and I_6 violate Eqs. (46), (49), and (51), and I_5 violates Eqs. (46–51). Only I_1, I_3, I_4, and I_7 fulfill Eqs. (46–51); therefore, the combined topological relation is

$$\text{meet} ; \text{coveredBy} \quad \Rightarrow \quad \text{meet} ; \text{inside} \ \vee \ \text{coveredBy} \ \vee \ \text{overlap}$$

5 All 64 Compositions of Binary Topological Relations between Spatial Regions without Holes

The 64 combined binary topological relations between spatial regions without holes (Table 1) were derived with a Prolog-like inference engine (10). Each intersection for the eight relations was expressed as a fact, while the combinations, Eqs. (20–27), translated into the rules deriving the intersections of the combined relations. Finally, a set of rules matched the intersections with the relations.

The analysis of the table reveals the following:

- All compositions of topological relations are valid, because none of the 64 compositions produces a contradicting intersection and all derived intersections match with at least one of the eight topological relations.
- equal is the identity relation, because the composition of any relation with equal results in the original relation.
- Besides the trivial compositions with equal, only twelve compositions are unique.
- Only the outcome of three compositions—(1) disjoint ; disjoint, (2) inside ; contains, and (3) overlap ; overlap—is fully undetermined, i.e., the combination of the constraints is such that none of the eight relations is excluded.
- Three relations are transitive—equal, inside, and contains—while for the other relations, the composition of the same relations may result in a different relation.
- Only two pairs of compositions are commutative: (1) coveredBy ; inside = inside ; coveredBy and (2) covers ; contains = contains ; covers.

Fig. 2 shows geometric examples for the derived relations. All combined topological relations could be realized among three 2-dimensional objects in $\mathbb{R}^2$.

6 Conclusions

A formalism was presented that derives the composition of two binary topological relations. It is based upon fundamental transitivity laws of point-sets and uses the 9-intersection representation, which is derived from the interiors, boundaries, and exteriors of the two target objects. The method enables qualitative spatial reasoning at a conceptually higher level than implementation-dependent data structures. For example, if two topological relations that connect a common object are known, or have been derived, then the direct relation may be inferred, even if no observations are available or only insufficient information exists to calculate it. The proposed methodology has the advantage over models of spatial relations based upon directions and distances (32, 38), because it uses only topological principles and, therefore, it can be implemented on a computer without struggling with the usual problems of the finiteness of the computer number systems. Likewise, the formalism is orientation-independent and, therefore, more general than a reasoning mechanism about spatial objects segmented into orthogonal symbolic projections (25, 27).

Now that the composition table for region-relations has been exhaustively determined, this knowledge can be used in spatial reasoning (34) and spatial query processing (15). For example, inconsistencies that are difficult to detect in complex topological queries, may be found prior to processing a query against a spatial database. Likewise, redundant constraints that are implied through a combination of other constraints may be eliminated before testing them against the database.

Future investigations will focus on the application of this formalisms to the relations between objects of different dimensions (e.g., between a region and a line), to objects with a codimension greater than 0 (e.g., two lines embedded in $\mathbb{R}^2$), and to objects with separated boundaries , such as regions with holes. This will contribute to a comprehensive topological reasoning system.

Acknowledgments

Renato Barrera, Andrew Frank, Christian Freksa, and Daniel Hernández provided helpful comments to an earlier version of this paper. Thanks also to Robert Cicogna for his editorial help in preparing this article.

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Table 1. The 64 compositions of the binary topological relations A r$_i$ B and B r$_j$ C (d = disjoint, m = meet, e = equal, i = inside, cB = coveredBy, ct = contains, cv = covers, and o = overlap).

	disjoint (B, C)	meet (B, C)	equal (B, C)	inside (B, C)	coveredBy (B, C)	contains (B, C)	covers (B, C)	overlap (B, C)
disjoint (A, B)	d ∨ m ∨ e ∨ i ∨ cB ∨ ct ∨ cv ∨ o	d ∨ m ∨ i ∨ cB ∨ o	d	d ∨ m ∨ i ∨ cB ∨ o	d ∨ m ∨ i ∨ cB ∨ o	d	d	d ∨ m ∨ i ∨ cB ∨ o
meet (A, B)	d ∨ m ∨ ct ∨ cv ∨ o	d ∨ m ∨ e ∨ cB ∨ cv ∨ o	m	i ∨ cB ∨ o	m ∨ i ∨ cB ∨ o	d	d ∨ m	d ∨ m ∨ i ∨ cB ∨ o
equal (A, B)	d	m	e	i	cB	ct	cv	o
inside (A, B)	d	d	i	i	i	d ∨ m ∨ e ∨ i ∨ cB ∨ ct ∨ cv ∨ o	d ∨ m ∨ i ∨ cB ∨ o	d ∨ m ∨ i ∨ cB ∨ o
coveredBy (A, B)	d	d ∨ m	cB	i	i ∨ cB	d ∨ m ∨ ct ∨ cv ∨ o	d ∨ m ∨ e ∨ cB ∨ cv ∨ o	d ∨ m ∨ i ∨ cB ∨ o
contains (A, B)	d ∨ m ∨ ct ∨ cv ∨ o	ct ∨ cv ∨ o	ct	e ∨ i ∨ cB ∨ ct ∨ cv ∨ o	ct ∨ cv ∨ o	ct	ct	ct ∨ cv ∨ o
covers (A, B)	d ∨ m ∨ ct ∨ cv ∨ o	m ∨ ct ∨ cv ∨ o	cv	i ∨ cB ∨ o	e ∨ cB ∨ cv ∨ o	ct	ct ∨ cv	ct ∨ cv ∨ o
overlap (A, B)	d ∨ m ∨ ct ∨ cv ∨ o	d ∨ m ∨ ct ∨ cv ∨ o	o	i ∨ cB ∨ o	i ∨ cB ∨ o	d ∨ m ∨ ct ∨ cv ∨ o	d ∨ m ∨ ct ∨ cv ∨ o	d ∨ m ∨ e ∨ i ∨ cB ∨ ct ∨ cv ∨ o

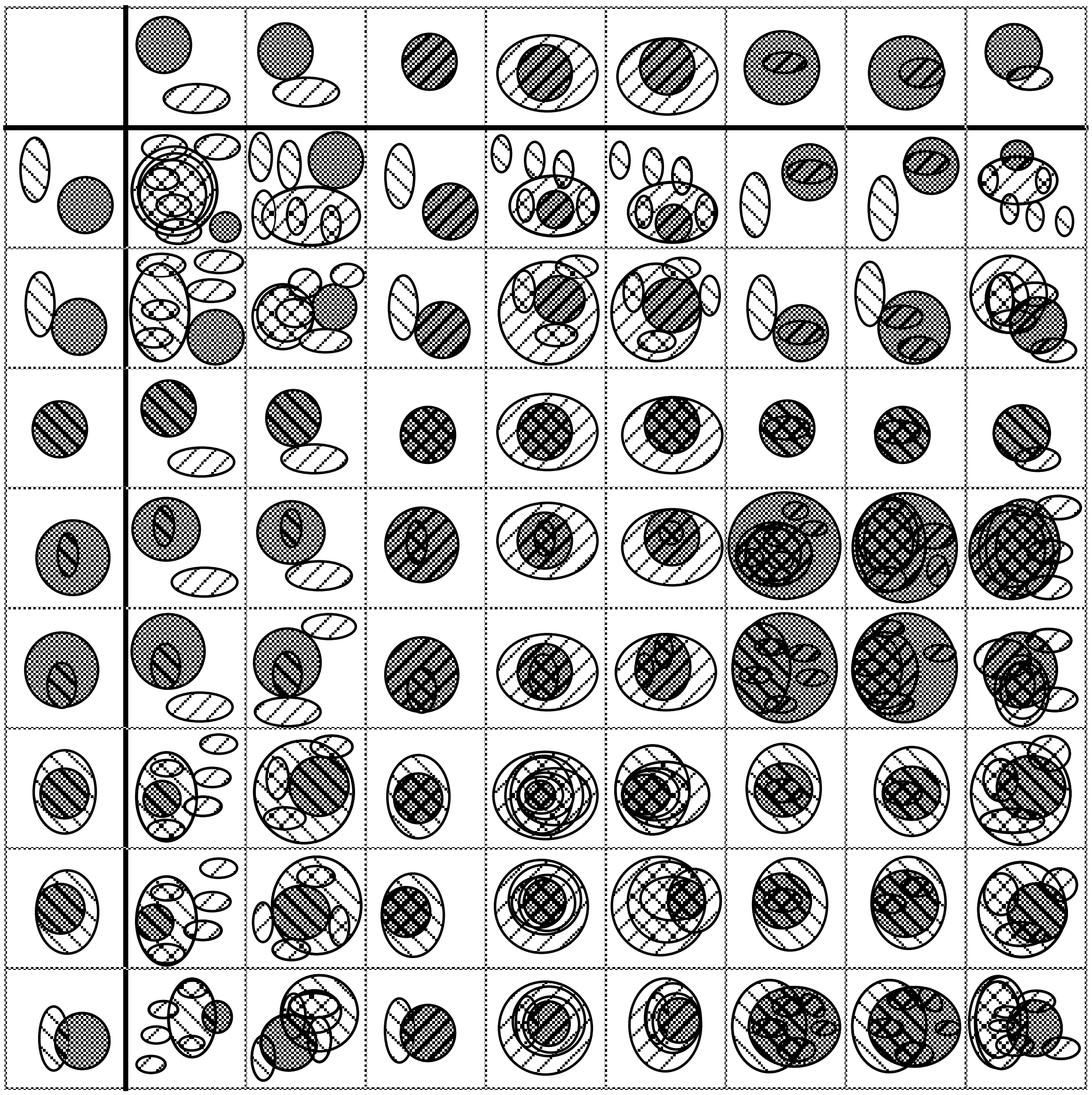

Figure 2. Geometric interpretations of the 64 compositions of binary topological relationships.